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P.B.R. Hazell and P.L. Scandizzo

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Competitive Demand Structures under Risk in Agricultural Linear Programming Models*

P. B. R. Hazell and P. L. Scandizzo

A method is presented for solving agricultural sector models under risk to obtain perfectly competitive levels of outputs and prices in all product markets when producers behave according to an E, V decision criterion. The nature of market equilibrium behavior is considerably more complicated under risk than in a deterministic setting. This presents difficulties in designing models which will always provide meaningful economic answers. These difficulties are overcome by stipulating conditions under which the proposed model is applicable. The resultant model is a quadratic programming problem, and linearization techniques are suggested which enable solutions to be obtained through conventional linear programming computer codes.

Key words: agricultural sector models; competitive markets; risk; linear programming.

Linear programming models are gaining increasing acceptance as tools for the sector analysis of agricultural supply response and agricultural investment programs. For many purposes it is desirable that such models provide the perfect competition solution to all product markets when both prices and quantities are endogenous. Samuelson [14] provided the basic methodology for achieving this in the deterministic case by utilizing the sum of consumers' and producers' surplus (net social product) as the model maximand. He developed this result in the context of spatial equilibrium models, and Takayama and Judge [17, 18] further developed this objective function to obtain a quadratic programming formulation for multiproduct models. Duloy and Norton [4, 5] subsequently applied the method to agricultural sector models using linear programming approximations.

The purpose of this paper is to provide a modification of the Duloy-Norton method when production is risky and individual farmers maximize E, V utility instead of expected profits. Neglect of such risk averse behavior can lead to important overstatements of the supply response of high risk enterprises, as well as overestimation of the returns to investment programs.

Since there are important interdependencies in the risk case between the way farmers behave and the nature of market equilibrium [8, 21], it will be necessary to explore both the micro and macro aspects of the model. The micro-macro aspects of the deterministic model are first reviewed, and then the implications of introducing risk are explored.

The Deterministic Model

The deterministic model is premised on the assumption that individual farmers are profit maximizers and that they compete in a perfectly competitive way. The latter assumption implies in particular that farmers plan on the basis of constant anticipated prices.

Define $\hat{p}$ = an $n \times 1$ vector of anticipated product prices,

c = an $n \times 1$ vector of unit costs,

x = an $n \times 1$ vector of enterprise levels,

M = an $n \times n$ diagonal matrix of enterprise yields with jth diagonal entry m_j,

and

$y = Mx$ is the $n \times 1$ vector of total outputs.

Then the objective function for an individual farm problem is

$$(1) \qquad \operatorname{Max} \pi = \hat{p}' y - c' x,$$

and this is to be maximized over some set of constraints which are usually specified to be linear.

If the product markets attain an equilib-

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Peter Hazell *and* Pasquale Scandizzo *are economists in the Development Research Center, International Bank for Reconstruction and Development, Washington, D. C.*

rium, then regardless of the way in which the anticipated prices $\hat{p}$ are formed over time, the equilibrium is unique. Furthermore, the market equilibrium prices and outputs occur at the points where the demand and implicit model supply functions intersect. This fact provides the basis of the solution procedure.

Let X, Y, C, and W be some appropriate aggregates[1] of the individual farm x, y, c, and M matrices, and P be the vector of unknown market prices. Then assuming the linear demand structure

$$(2) \qquad P = A - BY,$$

the Duloy-Norton aggregate model objective function is

$$(3) \quad \text{Max}\, \Pi = X'W\,(A - 0.5\,BWX) - C'X$$

where it is understood that $Y = WX$.

The term $X'W\,(A - 0.5\,BWX)$ is the sum of areas under the product demand functions. For example, in the single product case this would be

$$\int_0^y (a - by)\,dy = y(a - 0.5by)$$
$$= wx\,(a - 0.5bwx).$$

The term $C'X$ is total production costs, or equivalently, the sum of areas under the product supply functions. Consequently, the difference between these two terms is the sum of producers' and consumers' surplus over all markets, and this reaches its maximum at the required intersections of supply and demand functions.

Introduction of Risk

The basic source of risk to be introduced is confined to yields. Thus the vector of products for an individual farm now becomes $y = Nx$ where N is an $n \times n$ diagonal matrix of stochastic yields with jth diagonal element ϵ_j.

Stochastic yields imply stochastic supply functions and hence lead to stochastic market prices, P. It is assumed, however, that input costs and the market demand structure remain non-stochastic and that the farm linear programming constraints are not effected. The latter assumption can easily be relaxed, since several techniques are available to handle stochastic constraints that do not effect the farm model objective functions [2, 11, 13].

It is further assumed that the individual farmers are risk averse and that their behavior conforms to a single period E, V specification [9, 10, 15]. Consequently, the individual farm model objective function each year is

$$(4) \quad \text{Max}_{x}\, u = E(P'y) - c'x - \phi\, V(\hat{p}'y)$$

where E and V denote, respectively, the expectation and variance operators, and ϕ is a risk aversion coefficient.

To enumerate (4) more precisely, it is necessary to make explicit assumptions about the nature of farmers' subjective anticipations. These in part depend on the nature of perfect competition under risk.

Perfect Competition Under Risk

As a natural generalization of the deterministic concept of perfect competition, it is assumed that farmers continue to expect that their outputs do not have any effect on the market. A set of behavioral anticipations which are consistent with this for all j are the following:

A1 $E(\epsilon_j) = m_j$

A2 $V(\epsilon_j) = \sigma_{\epsilon_j}^2$

A3 $E(P_j) = \hat{p}_j$

A4 $V(P_j) = \sigma_{p_j}^2$

A5 $\text{Cov}(P_iP_j) = \sigma_{p_{ij}}$; $\text{Cov}(\epsilon_i\epsilon_j) = \sigma_{\epsilon_{ij}}$, all $i \neq j$

A6 $\text{Cov}(P_jy_i) = x_i\,\text{Cov}(P_j\epsilon_i) = 0$, all i,

where all operators are now subjective expectations and which may differ from the real world parameters. Assumption A3 states that farmers expect a constant mean price for each product, and by making the variance homoskedastic (A4) and the covariances between prices and outputs zero (A6), this implies that there is no expected relationship between the output of the individual farm and the market.

It is important to note that these are behavioral assumptions; it is not required that the farmer anticipate the true state of affairs. Market behavior is in part a reflection of what farmers anticipate, but this is basically no different from the deterministic case where, in the short run, the farmers' expectation of prices $\hat{p}$ can differ from the vector of market clearing prices P.

[1] Aggregation should be exact to avoid biased results in the sector model. The usual approach to this problem is through appropriate classification of farms into homogenous groups [3, 16].

Given the above set of assumptions, the components of (4) can be enumerated as follows.

$E(P'y) = \hat{p}'Mx$ where $M = E(N)$

$V(P'y) = x'\Omega x$ where Ω is an $n \times n$ covariance matrix of activity revenues with diagonal elements

$$\begin{aligned}
\omega_{jj} &= V(P_j\epsilon_j)\\
&= E(P_j^2\,\epsilon_j^2) - \hat{p}_j^2\, m_j^2\\
&= E(P_j^2)\,E(\epsilon_j^2) - \hat{p}_j^2\, m_j^2\\
&= \sigma_{p_j}^2\,E(\epsilon_j^2) + \hat{p}_j^2\,\sigma_{\epsilon_j}^2
\end{aligned}$$

and off-diagonal elements

$$\begin{aligned}
\omega_{ij} &= \text{Cov}(P_i\epsilon_i,\, P_j\epsilon_j)\\
&= E(P_iP_j\epsilon_i\epsilon_j) - E(P_i\epsilon_i)\,E(P_j\epsilon_j)\\
&= E(P_iP_j)\,E(\epsilon_i\epsilon_j) - \hat{p}_i\hat{p}_j\, m_im_j\\
&= E(P_iP_j)\,[E(\epsilon_i\epsilon_j) - m_im_j]\\
&\quad + m_im_j\,[E(P_iP_j) - \hat{p}_i\hat{p}_j]\\
&= [\sigma_{p_{ij}} + \hat{p}_i\hat{p}_j]\,\sigma_{\epsilon_{ij}}\\
&\quad + m_im_j\,\sigma_{p_{ij}}.
\end{aligned}$$

The farm problem objective function is then

$$(5) \qquad \text{Max}_{x}\, u = \hat{p}'Mx - c'x - \phi x'\Omega x.$$

Obviously, alternative assumptions to A1 through A6 are possible. However, providing A1, A3, and A6 are retained, then the only effect of changing the assumptions is on the elements of Ω. This will effect the market behavior and the estimation of Ω in a quantitative way, but does not deter the development of qualitative results about farm behavior.

Let the linear programming constraints for the farm model be denoted by $Dx \leqslant b$, then the Lagrangian function for maximizing (5) over this set is

$$(6) \quad L = \hat{p}'Mx - c'x - \phi x'\Omega x + v'(b - Dx)$$

where v is a vector of dual values. An optimal solution to the problem is then a saddle point, and necessary and sufficient conditions for any (x, v) to be this saddle point are obtainable from the Kuhn-Tucker conditions. The necessary conditions are

$$(7) \qquad \frac{\partial L}{\partial x} \leqslant 0, \frac{\partial L}{\partial v} \geqslant 0$$

$$(8) \qquad x\frac{\partial L}{\partial x} = 0,\ v\frac{\partial L}{\partial v} = 0.$$

Of these, the requirements in (8) are the complementary requirements that an activity cannot be active and at the same time have a non-zero opportunity cost, and a resource cannot be slack and at the same time have a non-zero dual value. Sufficient conditions for a saddle point can be derived but reduce to the requirement that Ω be a positive semidefinite matrix [18, p. 19].

Applying the necessary requirements in (7) to (6) gives

$$(9) \quad \frac{\partial L}{\partial x} = \hat{p}'M - c' - 2\phi x'\Omega - v'D \leqslant 0$$

$$(10) \qquad \frac{\partial L}{\partial v} = b - Dx \geqslant 0.$$

(10) is merely the feasibility requirement, but (9) contains the risk counterpart to the classical marginality rules for output determination in a deterministic firm. Taking the jth element of the vector $\frac{\partial L}{\partial x}$, rearranging terms, and dividing by m_j,

$$(11) \quad \hat{p}_j \leqslant \frac{1}{m_j}\,[\sum_k v_k\, d_{kj} + c_j + 2\phi \sum_i \omega_{ji}\, x_i]$$

This states that for each product the expected marginal cost per unit of output must be equal to or greater than the expected price. The expected marginal cost comprises expected own marginal cost c_j/m_j plus a marginal risk factor $\frac{\phi}{m_j}\frac{\partial V(P'y)}{\partial x_j} = \frac{1}{m_j}\, 2\phi \sum_i \omega_{ji}\, x_i$, plus expected opportunity costs $\frac{1}{m_j}\sum_k v_k\, d_{kj}$ as reflected in the dual values of the resources used by that activity. This differs from the comparable requirements of a deterministic model primarily in that a risk term has been introduced. This is quite reasonable because the risk term is really nothing but a new cost, namely, the additional expected return demanded by farmers as compensation for taking risk. This is even clearer when farmers can participate in a crop insurance program, for then the risk term is the marginal premium a farmer would be willing to pay to insure against risk, i.e., a certainty equivalent cost.

This result is not new and is consistent with the results obtained for single product firms in economic analysis (e.g., see [12]). Further, the appearance of the risk factor as a marginal cost provides the rationale for the expectation that deterministic models overestimate the supply response of high risk crops. This is because $\Sigma_j\omega_{ji}x_i$ will then be positive, hence the marginal cost curve must lie above the marginal cost curve that would be obtained from a deterministic (or risk-neutral) model.

While (11) is a necessary condition, it is clear from duality theory that the condition will always be satisfied as an equality in an optimal solution for all activities which enter the basis [18, ch. 2]. Consequently, for all basic activities the risk counterpart to the price-equals-marginal-cost rule can be written as:

$$(12)\quad \hat{p}_j = \frac{1}{m_j}\,[\sum_k v_k d_{kj} + c_j + 2\phi \sum_i \omega_{ji} x_i].$$

The right hand side of (12) is then the short-run supply function for the farm as implicitly embedded in the programming model. This is a basic behavioral relationship and expresses the farmer's determination of x_j given his expectations about yields and prices. That is, $x_j = f(M, \hat{p}, \Omega)$, everything else constant. Multiplying by the mean yield m_j, a conditionally expected supply function is immediately obtained:

$$(13)\quad E(y_j|x_j) = m_j x_j = m_j f(M, \hat{p}, \Omega).$$

Since all the expectations involved are subjective anticipations, it is useful to denote (13) as the *anticipated* supply function for the farm to distinguish it from a true statistical relationship.

Market Equilibrium Under Risk

By summing the anticipated supply functions over all farms, an aggregate anticipated supply function can be obtained as a basic behavioral relationship in the market. Ignoring aggregation problems for now, the jth supply function can be written as

$$(14)\quad E(Y_j|X_j) = w_j X_j = w_j\, g(W, \rho, \Gamma)$$

where X_j, Y_j, W, $\hat{p}$, and Γ are suitable aggregates of x_j, y_j, M, $\hat{p}$, and Ω respectively, and w_j is the jth diagonal element of W.

Given X_j, actual supply is

$$(15)\quad Y_j|X_j = e_j X_j = e_j g(W, \hat{p}, \Gamma)$$

where e_j is a suitable aggregate of the farm ϵ_j's such that $E(e_j) = w_j$. Clearly, actual supply is stochastic with e_j and, furthermore, is of a specification in which the slope of the supply function is stochastic. Ignoring the unnecessary complication of a stochastic intercept term and assuming e_j to be bounded on some positive interval $e_m \leqslant e_j \leqslant e_x$, the market solution can be portrayed as in Figure 1.

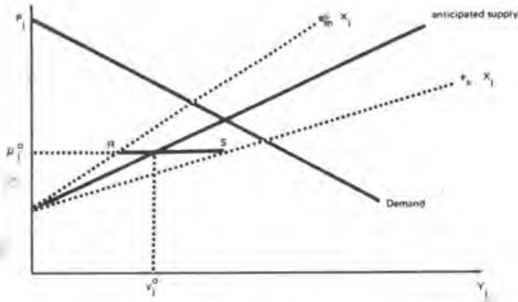

Figure 1

If W and Γ remain constant, the anticipated supply function is fixed, and aggregate anticipated supply is determined by $\hat{p}$. Thus for given $\hat{p}$ with $\hat{p} = \rho_j^o$, farmers will plan their farms so that aggregate expected output is Y_j^o. However, because yields are stochastic over the range e_m to e_x, actual supply can take on any value between R and S. More generally, the actual supply function can rotate around the anticipated supply function to any position contained in the funnel defined by $e_m X_j$ and $e_x X_j$. It follows that market price must always be stochastic and will fluctuate with both e_j as well as with X_j if the latter does not stabilize to some equilibrium amount. Further complications arise when W or Γ are not fixed, for then the whole "supply funnel" may shift structurally over time.

The question now arises as to what is a perfectly competitive and equilibrium solution to the market, and since it is obviously not a point solution, what characteristics can usefully be derived with a mathematical programming model? The intuitive answer would be to view the market as stabilizing in its price distribution and to seek the perfectly competitive solution for expected price and output at the intersection of the demand and anticipated supply schedules. Unfortunately, while equilibrium is appropriately viewed in terms of a stabilizing price distribution, the properties of this distribution do depend on the way in which farmers form their expectations about $\hat{p}$, W, and Ω. That is, the equilibrium is interdependent with the adjustment mechanism. Further, only under certain conditions do the demand and anticipated supply schedules actually intersect at the equilibrium value of expected price. It is therefore necessary to explore these problems in greater depth.

To initiate the analysis, the following assumptions are made:

- A7 W and Γ are fixed.
- A8 X_j is a linear function in $\hat{p}_j$ of the form $X_j = \lambda \hat{p}_j$ where λ is some appropriate function of W and Γ and is therefore a constant by A7. That is, the response function for X_j in a programming model is being approximated by a straight line. Actual supply in the tth year is then $Y_{jt} = \lambda e_{jt} \hat{p}_{jt}$.
- A9 Demand is linear and of the form $P_{jt} = a - bY_{jt}$.
- A10 Farmers form their anticipated, or mean, price forecasts each year as a weighted average of past prices.[2] That is,

$$(16)\quad \hat{p}_{jt} = \sum_{i=1}^{m} \gamma_i P_{jt-i} \text{ and } \sum_{i=1}^{m} \gamma_i = 1.$$

  Note that a naive cobweb formulation and the Nerlove type adaptive expectation models are special cases of (16), so that the formulation is quite general.
- A11 $\mathrm{Cov}(e_{jt}, e_{jt-1}) = 0$, all t. That is, the yield of an individual activity is uncorrelated with itself over time.

Ignoring the jth subscript for convenience, it is evident that if the market clears each year, then market clearing price is

$$(17)\quad P_t = a - b\,\lambda e_t\, \hat{p}_t = a - b\,\lambda e_t \sum_{i=1}^{m} \gamma_i P_{t-i}.$$

Equilibrium can now be defined in terms of the convergent properties of P_t over time. There are a number of alternative properties to choose from [21], but basically all are variants of the concept of convergence in the probability density function of P_t and its various central moments. Consider first the convergence of expected price:

$$\begin{aligned} E(P_t) &= a - b\,\lambda\, E(e_t\, \hat{p}_t) \\ (18)\qquad &= a - b\,\lambda\, w\, E(\hat{p}_t) \\ &= a - b\,\lambda\, w \sum_{i=1}^{m} \gamma_i E(P_{t-i}). \end{aligned}$$

This is an mth order difference equation, and if convergence occurs, has the particular solution

$$(19)\quad \lim_{t\to\infty} E(P_t) = \frac{a}{1 + b\,\lambda\, w}.$$

Further, it is the same for any choice of γ weights satisfying (16).

Solving now for the intersection price (P^*) of demand and anticipated supply ($Y_t|X_t = \lambda w_t\ \hat{p}_t$), $\frac{a}{b} - \frac{1}{b}\ P^* = \lambda\ w\ P^*$; hence $P^* = \frac{a}{1 + b\,\lambda\, w}$ which is identical to $\lim E(P_t)$.

Thus, under assumptions A7 through A11, the asymptotic expectation of market price is the same regardless of the specific price learning model, and, furthermore, corresponds to the desired intersection of the demand and anticipated supply functions. It is also clear from the derivation of anticipated supply that at this point $\lim E(P_t) = E(\text{marginal costs})$, which provides an acceptable equivalence to the equilibrium point of a competitive but deterministic market.

Turning now to other properties of the equilibrium, it can be shown that even under assumptions A7 through A11, the variance and probability density function of price *do not* converge to the same limits for alternative choices of the γ weights [8]. These properties of the equilibrium do depend on the way farmers form their price anticipations each year and consequently can only be enumerated through simulation type procedures given explicit assumptions about the behavior involved. A similar result pertains for the stochasticity of input decisions, and hence X_j.

The results for $E(P_t)$ are clearly quite useful, but what happens if any of the assumptions A7 through A11 are relaxed?

(a) If either A8 or A9 are relaxed to permit non-linearities, then $\lim E(P_t)$ usually becomes dependent on the γ weights in (16) and no longer corresponds to the intersection of demand and anticipated supply. Under these conditions the intersection price is only a linear approximation to the asymptotic expectation of price.

(b) Relaxation of A7 leads to a situation in which the slope of the supply function (λ) varies structurally over time, and the location of the anticipated supply function will no longer be fixed. However, if W and Γ converge,[3] then a stable equilib-

[2] It is to be noted that an E, V decision specification only states which price and yield parameters are relevant each year; it does not state how anticipations about these parameters are formed over time.

[3] It is interesting to note that if assumptions A1 through A6 are a true description of competitive behavior, then Γ is unlikely to converge to its real statisti-

rium is attained with the above properties, and the only effect is on the speed of attaining the equilibrium. A similar result pertains when A10 is relaxed providing $\lim E(\hat{p}_t) = \lim E(P_t)$; otherwise approximations are again involved.

(c) A11 is only a simplifying assumption and can be relaxed to consider autocorrelated yields providing the stochastic residual of the process satisfies assumption A11. The main result is an increased complexity in the algebra.

The authors feel that assumptions A7 through A11 and the possible modifications stated above provide quite reasonable approximations to the real world and that the analytical results derived therefrom provide a reasonable description of market equilibrium behavior. On this assumption, any mathematical programming models that could provide the intersection solution for demand and anticipated supply would generate results that have a direct and relevant economic interpretation and that would be reasonably general with respect to the way in which farmers form their price and yield anticipations over time.[4] The task remains of providing a modification to the Duloy-Norton objective function that solves this problem in an aggregate model.

Solving for the Asymptotic Expectation of Price

X, Y, W, C, $\hat{\rho}$, and Γ have already been defined as aggregates of the farm x, y, M, c, $\hat{p}$ and Ω matrices. Also, let Φ be an aggregate of the farm risk parameters ϕ.

In forming these aggregates, there are definite problems that are closely related to the problem of establishing farm classification criteria for exact aggregation in quadratic models. This problem lies beyond the scope of this paper, and it is merely noted that the aggregate variables Γ and Φ must be chosen so that

$$\Phi X'\Gamma X = \sum_k \phi_k x'_k \Omega_k x_k$$

where k denotes the kth individual farm. Without this condition any possible covariance relationships between farms could be exploited in the aggregate model in seeking efficient diversification, and this would be inconsistent with assumed competitive behavior. Thus, for example, if all the farms were identical, a suitable choice of the aggregate variables would be $\Phi = \frac{1}{k}\phi$ and $\Gamma = \Omega$ so that $\sum_k \phi_k x'_k \Omega_k x_k = K \phi x' \Omega x = K^2 \Phi x' \Gamma x = \Phi X'\Gamma X$ where $X = Kx = \sum_k x_k$.

Given the necessary aggregate variables and parameters, it is now possible to modify the Duloy-Norton objective function to obtain the solution corresponding to the intersection of demand and anticipated supply schedules. This modified function is

$$\text{(21)} \quad \text{Max } U = X'W(A - 0.5BWX) - C'X - \Phi X'\Gamma X$$

where $X'W(A - 0.5\ BWX)$ is now the sum of expected areas under the demand curves and $C'X + \Phi X'\Gamma X$ is a revised sum of areas under the supply curves.

To verify that (21) gives the desired solution, form the Lagrangian function $L = X'W(A - 0.5\ BWX) - C'X - \Phi X'\Gamma X + v'(b - DX)$ where b and D denote the aggregate constraints and v is a vector of dual values. Apart from complementary requirements, the necessary Kuhn-Tucker conditions are:

$$\text{(22)} \quad \frac{\partial L}{\partial X} = WA - WBWX - C - 2\,\Phi\Gamma X - D'v \leqslant 0$$

and

$$\text{(23)} \quad \frac{\partial L}{\partial v} = b - DX \geqslant 0.$$

(23) is the feasibility requirement, and (22) can be rearranged as

$$\text{(24)} \quad (A - BWX) \leqslant W^{-1}[C + 2\Phi\Gamma X + D'v].$$

Now WX is the vector of anticipated supplies $E(Y|X)$, and $A - BWX$ is the corresponding vector of market prices. Further, the right-hand side of (24) is the sum of expected marginal cost curves over all farms [the jth component is the sum of the right hand sides of equations like (11)]. That is, it is the vector of aggregate anticipated supply functions. The inequality in (24) states that in aggregate, farmers must operate around some expected point on the anticipated supply functions which lie at or

cal value, for there will almost certainly be non-zero correlations between individual farm outputs and market prices.

[4] Extension of the results for $E(P_t)$ to the multi-product case are quite straightforward, but are too lengthy to report here.

above the intersections with demand. Clearly, by duality, optimality occurs at the intersection point for all non-zero activities in the solution, and then $A - BWX$ is the intersection price vector. However, conditions have already been established for this to be an approximation to the vector of asymptotically expected prices, and if these conditions are met, the perfectly competitive solution for $\lim E(P) = E$ (marginal costs) will have been obtained.

Linear Programming Approximations

The aggregate model with the objective function defined in (21) is a quadratic programming problem. Because of the large dimensions of any realistic sector model and the difficulties that still exist with quadratic programming computer codes, it is clearly desirable to linearize this problem.

Duloy and Norton [4, 5] have shown how the term $Y'(A - 0.5\ BY)$ where $Y = WX$ can be linearized. To illustrate their method, consider the simplest case when B is diagonal, implying that the product demands are independent. Then, letting V_j denote the area under the demand curve (the definite integral) from 0 to Y_j for the jth product, $Y'(A - 0.5\ BY) = \sum_{j=1}^{n} V_j$.

V_j is a quadratic, concave function when plotted against Y_j, and since the programming model is a maximization problem, V_j can be approximated by a series of linear steps using conventional linear programming computer codes. Duloy and Norton introduce additional activities, V_{ij}, i = 1 to k for each V_j; assign upper bounds, $V_{ij} \leqslant g_{ij}$ on Y_j over which interval V_{ij} applies; and assign a single value of V_j, say, d_{ij}, which is to approximate V_j over the interval $Y_j \leqslant g_{ij}$. They then suggest that the part of the programming problem involving Max $Y'(A - 0.5\ BY)$ with $Y = WX$ be replaced by the LP problem.

$$\text{(24)} \quad \text{Max} \sum_j \sum_{i=1}^{k} d_{ij} V_{ij}$$

such that

$$\text{(25)} \quad X_j m_j - \sum_i g_{ij} V_{ij} \geqslant 0, \qquad \text{all } j$$

$$\text{(26)} \quad \sum_i V_{ij} \leqslant 1, \qquad \text{all } j.$$

This method adds only two rows for each product, but permits inclusion of as many V_{ij} activities as desired to increase the accuracy of the approximation to any degree of precision.

The remaining quadratic term $\Phi X'\Gamma X$ can be linearized along the lines suggested by Thomas *et al.* [19]. However, the approach leads to large programming problems that must be solved with special separable LP algorithms and that do not necessarily yield a global optimum solution. Alternatively, if Γ is estimated on the basis of time series data, the mean absolute deviation (*mad*) method proposed by Hazell [7] can be used.

Let $r_{jt} = P_{jt}m_{jt}$ denote the tth observation, $t = 1$ to T, on the revenue of the jth activity X_j, and let $\bar{r}_j$ denote the sample mean revenue for the activity over the T years.[5] Then, following Hazell, the classical estimate of variance used in the programming model

$$\text{(27)} \quad \text{EST}(X'\Gamma X) = \sum_i \sum_j X_i X_j \left[\frac{1}{T-1} \sum_t (r_{jt} - \bar{r}_j)(r_{it} - \bar{r}_i) \right]$$

can be replaced by the less efficient, but more easily linearized estimator

$$\text{(28)} \quad \text{EST}(X'\Gamma X) = \Delta \left\{ \frac{1}{T} \sum_t \Big| \sum_j (r_{jt} - \bar{r}_j) X_j \Big| \right\}^2$$

where

$\Delta = T\pi/2(T-1)$ and

$\pi =$ the mathematical constant.

By defining new variables $z_t \geqslant 0$, all t, which measure negative deviations in total revenue around the mean for the tth set of revenue outcomes [7], it follows that

$$2 \sum_t z_t = \sum_t \Big| \sum_j (r_{jt} - \bar{r}_j) X_j \Big| \qquad (= Z).$$

To obtain the estimated variance, it merely remains to approximate the square of Z using the Duloy and Norton method discussed above and then to multiply the resultant value of Z^2 by Δ/T^2. Define variables V_i, $i = 1$ to k, with upper bounds $V_i \leqslant g_i$, and assign values d_i to each V_i which approximate the value of Z^2 over

[5] The raw data should first be analyzed for any trend and other systematic movements over time, and these components removed to obtain a random residual.

Table 1. Layout of tableau for the linearized problem

Constraint and Text Equation Numbers	Production Activities X_1	$X_2 \ldots\ldots X_n$	n Sets of Activities to Linearize Areas Under Demand Functions $V_{11} \ldots V_{1k}$	$V_{21} \ldots V_{2k} \ldots V_{n1} \ldots V_{nk}$	T Negative Deviation Counters $z_1 \ldots\ldots z_T$	Sum Negative Deviations Z	k Activities to Linearize Z^2 $V_1 \ldots\ldots V_k$	RHS
Objective Function (21), (24) and (29)	$-c_1$	$-c_2 \ldots\ldots -c_n$	$d_{11} \ldots d_{1k}$	$d_{21} \ldots d_{2k} \ldots d_{n1} \ldots d_{nk}$	$0 \ldots\ldots 0$	0	$-(\Phi\Delta/T^2)d_1 \ldots -(\Phi\Delta/T^2)d_k$	Maximize
n Commodity Balance Constraints (25)	m_1	$m_2 \ldots m_n$	$-g_{11} \ldots -g_{1k}$	$-g_{21} \ldots -g_{2k} \ldots -g_{n1} \ldots -g_{nk}$				$\geqq 0$
n Convex Combination Constraints (26)			$1 \ldots 1$	$1 \ldots 1 \ldots 1 \ldots 1$				$\leqq 1$
T Revenue Constraints Containing Sample Data (30)	$r_{11} - \bar{r}_1 \ldots r_{1T} - \bar{r}_1$	$r_{21} - \bar{r}_2 \ldots r_{n1} - \bar{r}_n \ldots r_{2T} - \bar{r}_2 \ldots r_{nT} - \bar{r}_n$			$1 \ldots 1$			$\geqq 0$
Z Identity (31)					$2 \ldots\ldots 2$	-1		$= 0$
Z^2 Balance (32)						1	$-g_1 \ldots\ldots -g_k$	$\leqq 0$
Convex Combination (33)							$1 \ldots\ldots 1$	$\leqq 1$
Resource Constraints	D Matrix							$\leqq$ b vector

the interval $Z \leqslant g_i$. Then the problem of minimizing $X'TX$ can be approximated by:

$$\text{(29)} \qquad \text{Min}(\Delta/T^2) \sum_{i=1}^{k} d_i V_i$$

such that

$$\text{(30)} \qquad \sum_j (r_{jt} - \bar{r}_j) X_j + z_t \geqslant 0, \text{ all } t.$$

$$\text{(31)} \qquad 2 \sum_t z_t = Z$$

$$\text{(32)} \qquad Z - \sum_i g_i V_i \leqslant 0$$

$$\text{(33)} \qquad \sum_i V_i \leqslant 1.$$

Results about the reliability of this approach compared to using (27) are available elsewhere [7, 20].

In order to show how all these approximations fit together, a complete LP tableau is formulated in Table 1 that approximates the solution to the original quadratic programming problem. If all the quadratic terms are approximated by k segments, the linearized problem adds an additional $k(n + 1) + T + 1$ activities and $n + T + 3$ rows to the problem, but this is highly efficient computationally when solutions are obtained through a revised simplex algorithm.

Conclusions

In this paper a methodology has been developed for simulating, with an LP model, the market equilibrium of a perfectly competitive but risky agriculture in which producers behave according to an E, V decision criterion.[6] Such results are useful for comparative static analysis of policy problems, and, according to the degree of risk involved, should be more descriptive than existing models which ignore risk.

The development of this methodology has pinpointed a number of difficult issues with respect to both the design and implementation of aggregate risk models.

First, market equilibrium is considerably more complicated under risk than in a deterministic setting, with interactive effects between the way farmers form their anticipations about prices and yields and the properties of an equilibrium if attained. Consequently, it is difficult to design a general programming model which will always provide meaningful economic answers. In this paper the problem was resolved by specifying a set of plausible assumptions under which the proposed model is appropriate but with an obvious loss in generality.

Second, an aggregate model must be defined in terms of variables which are inherently difficult to measure. This is partly because they are based on individual farmer's utility functions and subjective expectations about stochastic variables and are therefore difficult to observe, but partly because they involve aggregation procedures which have not been adequately explored in the literature. The authors have overcome these problems in practice by using statistical estimates of the aggregate variables obtained from time series data. While such results have been seemingly useful, the variables used are at best proxies for their real counterparts, and the method leaves much to be desired as a research technique.

These problems ought to be explored more fully given the importance of risk to agricultural planners throughout the world.

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[6] The model can readily be adapted to handle situations in which alternative decision criteria are appropriate. For example, if producers behave according to an E, standard deviation criterion [1, 6], then the aggregate model maximand should be $\text{Max } U = X'W\,(A - 0.5\,BWX) - C'X - \Phi(X'TX)^{1/2}$. Mixtures of decision criteria are also possible, but an economic interpretation of the market results is then more complex.

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