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Fixity of Volumes (shares) approach as Least square deviations (LSQ) and as the Least absolute deviations (LAD) procedures

The International Comparisons Program (ICP) "produces" the purchasing power of currencies and real output (volumes) of the countries in the world. The ICP is broken up into six regions and the regional comparisons are carried out firstly. Global results are then obtained by linking these regions together. A requirement in the ICP is that the global results (at basic heading level and at higher levels of aggregation) satisfy within-region fixity. It means the parities or volume indices of a pair of countries from the same region should be the same in the global comparison as in the within-region comparison [see ICP Manual, Chapter 15].

What indicator should be fixed – PPP or Volumes? The analysis [S. Sergeev (2005) and R. Hill (2010), page 14][1] showed that the results obtained by these different approaches are not identical. Obviously, this question concerns only inter-regional comparisons because it is sufficient to fix any indicator (PPP or volume indices) in the intra-regional comparisons and other indicator will be fixed automatically due to the equality : $I_{exp} = I_{pr} * I_{vol}$ [expenditure ratio is the product of price ratio (PPP) and volume ratio].

The detailed analysis of fixity for the PPPs (this can be presented as unweighted geometric scaling)[2] can be found in ICP Manual, Chapter 15.

R. Hill (2010) demonstrated in an explicit form that the fixity of PPPs can be presented as an optimization by altering the multilateral PPP by the minimum least-squares amount necessary to ensure that within-region fixity is satisfied. The fixity of PPPs is used in the present time in the Eurostat-OECD comparisons. It is possible to demonstrate that the main sense and content of the Eurostat-OECD approach is absolutely the same as the approach proposed by R. Hill (the techniques are some different but the fixed PPPs are the same).

The **fixity of volumes ratios** was used earlier in the ICP when the G-K method was used for the aggregation – the Regional volumes obtained during the global aggregation are redistributed in accordance with the shares obtained in the Regional comparisons [see Heston (1986) and Dikhanov (2007)])[3]. Some TAG members were interested - **Is it possible to prove that this method minimizes the changes needed to real expenditures to maintain fixity?** A respective attempt is done below.

To avoid the problems with measuring currency units and with the base for the Volume indices, it is better to operate not with country's real expenditure but with

[1] In the PPP aggregation software prepared many years ago for ESTAT by the author of this notice (this software was used also in the CIS, African, ESCWA and LAC 2011 ICP rounds) this choice is an option for users.

[2] The fixity of PPP (unweighting geometric scaling) is the present Eurostat-OECD approach. The concrete Eurostat-OECD procedure is not fully symmetrical because the fixity is needed only for the EU and OECE countries but not for other country's sub-sets. However the main sense and content of the Eurostat-OECD approach is absolutely the same as the approach proposed by R. Hill.

[3] This procedure can be presented also as the use weighted harmonic scalars for the PPPs (ICP Manual, Ch.15).

country's shares in real expenditures. This does not change the task principally because the ratios of shares in real expenditures are the same as ratios of real expenditure, i.e. all volume indices are not changed.

Let us have 3 Regions: Region A (no. of countries Na), Region B (no. of countries Nb) and Region C (no. of countries Nc).

Let **$Sh(A_j)$_GUnfix** – share of country j from Region A in the World real expenditure obtained in the unfixed Global comparison [**$Sh(B_k)$_GUnfix** and **$Sh(C_l)$_GUnfix** are the same countries for countries k and l in regions B and C).

The volumes shares within the Global comparison (all countries from all Regions simultaneously)[4] obtained without fixity can be written as follows:

$$Sh(A_j)\_GUnfix = \mathbf{Sh\_GUnFix\_R_A} * Sh(A_j)\_GUnFix\_R$$

$$Sh(B_k)\_GUnfix = \mathbf{Sh\_GUnFix\_R_B} * Sh(B_k)\_GUnFix\_R$$

$$Sh(C_l)\_GUnfix = \mathbf{Sh\_GUnFix\_R_C} * Sh(C_l)\_GUnFix\_R$$

where

$Sh\_GUnFix\_R_A$ – share of Region A in the World real expenditure obtained in the unfixed Global comparison

$Sh(A_j)$_GUnFix_R – share of country j from Region A in the Regional real expenditure obtained in the unfixed Global comparison

To have the fixity of Volume shares, we should take into account the results of Regional comparisons as the constraints. The within-region fixity for Volumes shares constraints can be written as follows[5]:

$$Sh(A_j)\_GFix = \mathbf{Sh\_R_A} * Sh(A_j)\_R$$

$$Sh(B_k)\_GFix = \mathbf{Sh\_R_B} * Sh(B_k)\_R$$

$$Sh(C_l)\_GFix = \mathbf{Sh\_R_C} * Sh(C_l)\_R$$

where

$Sh(A_j)$_GFix – share of country j from Region A in the World real expenditure obtained in the Global comparison with fixity (similar parameters for Regions B & C)

Sh_R_A – (unknown) share of Region A in the World real expenditure obtained in the Global comparison with fixity (similar parameters for Regions B and C)

$Sh(A_j)$_R – share of country j from Region A in the Regional real expenditure obtained in the original Regional comparison (similar parameters for Regions B & C)

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[4] The shares of a country in Real expenditure of a region or the World are nothing more as Volume indices "Country / Region" or "Country / World". The shares of a Region in Real expenditure of the World are nothing more as Volume indices "Region / World". Therefore the product of the share of a Region in the World and the share of a country within this Region is Volume index "Country / World":

VI "Region / World" x VI "Country / Region" = VI "Country / World".

[5] These constraints are some analogues of the R.Hill constraints for PPPs – see equations (4)-(6) in Hill (2010). Sh_R_A, Sh_R_B, Sh_R_C – analogues of the parameters α, β, γ in R.Hill approach.

If we want to obtain the final volume shares in the global world comparison in such way that these (in total) are deviate minimally (possible) from volume shares in the original Global comparison (all countries simultaneously) and simultaneously to keep the fixity of the regional shares then we should choice the values of variables **Sh_R_A**, **Sh_R_B**, **Sh_R_C** which **minimize the sum of the deviations.** One can use the sum of least squares of deviations as well as the sum of absolute deviations. Both procedures are described below.

Fixity of Volumes (shares) as the Least absolute deviations (LAD) procedure

$$_{Na}\Sigma|Sh(A_j)\_GUnfix - Sh(A_j)\_GFix| + {}_{Nb}\Sigma|Sh(B_k)\_GUnfix - Sh(B_k)\_GFix| + {}_{Nc}\Sigma|Sh(C_l)\_GUnfix - Sh(C_l)\_GFix| \Rightarrow \mathbf{min}$$

Or in more expanded form:

$$_{Na}\Sigma|\mathbf{Sh\_GUnFix\_R_A} * Sh(A_j)\_GUnFix\_R - \mathbf{Sh\_R_A} * Sh(A_j)\_R| + {}_{Nb}\Sigma|\mathbf{Sh\_GUnFix\_R_B} * Sh(B_k)\_GUnFix\_R - \mathbf{Sh\_R_B} * Sh(B_k)\_R| + {}_{Nc}\Sigma|\mathbf{Sh\_GUnFix\_R_C} * Sh(C_l)\_GUnFix\_R - \mathbf{Sh\_R_C} * Sh(C_l)\_R| \Rightarrow \mathbf{min}$$

We should find **Sh_R_A**, **Sh_R_B**, **Sh_R_C** by which the term (...) has minimal value. Simultaneously there is an additional condition:

$$\mathbf{Sh\_R_A} + \mathbf{Sh\_R_B} + \mathbf{Sh\_R_C} = 1$$

The minimization of the sum of absolute deviation for shares in the combination with the constraint above is a task for Nonlinear programming with the restriction(s). This can be solved by the method of Lagrange multipliers. The solution should bring the following values for unknown variables **Sh_R_A**, **Sh_R_B**, **Sh_R_C**:

$$\mathbf{Sh\_R_A} = \mathbf{Sh\_GUnFix\_R_A}$$

$$\mathbf{Sh\_R_B} = \mathbf{Sh\_GUnFix\_R_B}$$

$$\mathbf{Sh\_R_C} = \mathbf{Sh\_GUnFix\_R_C}$$

Fixity of Volumes (shares) as the Least square deviations (LSQ) procedure

The fixity of the Volume indicators can be presented in the terms of minimization of square of deviations of volume shares - as weighted LSQ procedure. Let us assume that we want to minimize the sum of the square deviations between country's shares in Word real expenditure obtained without and with fixity. This can be written mathematically in the following form:

Let **$Sh(A_j)$_GUnfix** – share of country j from Region A in the World real expenditure obtained in the unfixed Global comparison [**$Sh(B_k)$_GUnfix** and **$Sh(C_l)$_GUnfix** are the same countries for countries k and l in regions B and C).

To have the fixity of Volume shares, we should take into account the results of Regional comparisons as the constraints. The within-region fixity for Volumes shares constraints can be written as follows[6]:

$$Sh(A_j)\_GFix = \mathbf{Sh\_R_A} * Sh(A_j)\_R$$

$$Sh(B_k)\_GFix = \mathbf{Sh\_R_B} * Sh(B_k)\_R$$

$$Sh(C_l)\_GFix = \mathbf{Sh\_R_C} * Sh(C_l)\_R$$

where

$Sh(A_j)$_GFix – share of country j from Region A in the World real expenditure obtained in the Global comparison with fixity (similar parameters for Regions B & C)

Sh_R_A – (unknown) share of Region A in the World real expenditure obtained in the Global comparison with fixity (similar parameters for Regions B and C)

$Sh(A_j)$_R – share of country j from Region A in the Regional real expenditure obtained in the original Regional comparison (similar parameters for Regions B & C)

If we want to obtain the final volume shares in the global world comparison in such way that these (in total) are deviate minimally (in the terms of square deviation) from volume shares in the original Global comparison (all countries simultaneously) and simultaneously to keep the fixity of the regional shares then we should choice the values of variables **Sh_R_A**, **Sh_R_B**, **Sh_R_C** which **minimize the sum of the square deviations**:

$$_{Na}\Sigma(Sh(A_j)\_GUnfix - Sh(A_j)\_Gfix]^2 + {}_{Nb}\Sigma[Sh(B_k)\_GUnfix - Sh(B_k)\_Gfix]^2 + + {}_{Nc}\Sigma[Sh(C_l)\_GUnfix - Sh(C_l)\_Gfix]^2 \mathbf{\Rightarrow min}$$

To equalize the impact of the countries with higher and lower shares within the Regions, the weighting is desirable. **The weighting inversly proportional of shares of Regional shares of countries** - $Sh(A_j)$_R, $Sh(B_k)$_R, $Sh(C_l)$_R seems to be appropriate for this purpose.

So, the final function for the minimization is the following:

$$_{Na}\Sigma[Sh(A_j)\_GUnfix - \mathbf{Sh\_R_A} * Sh(A_j)\_R]^2/Sh(A_j)\_R + + {}_{Nb}\Sigma[Sh(B_k)\_GUnfix - \mathbf{Sh\_R_B} * Sh(B_k)\_R]^2/Sh(B_k)\_R + + {}_{Nc}\Sigma[Sh(B_k)\_GUnfix - \mathbf{Sh\_R_c} * Sh(C_l)\_R]^2/Sh(C_l)\_R \mathbf{\Rightarrow min}$$

Solving this relatively **Sh_R_A**, **Sh_R_B**, **Sh_R_C**, the following 1st order conditions are obtained:

$$-2* {}_{Na}\Sigma[Sh(A_j)\_GUnfix * Sh(A_j)\_R)/Sh(A_j)\_R] + 2*\mathbf{Sh\_R_A} {}_{Na}*\Sigma[Sh(A_j)\_R^2/Sh(A_j)\_R] =0$$

$$-2*{}_{Na}\Sigma[Sh(B_k)\_GUnfix*Sh(B_k)\_R)/Sh(B_k)\_R] + 2*\mathbf{Sh\_R_B} {}_{Na}*\Sigma[Sh(B_k)\_R^2/Sh(B_k)\_R] = 0$$

$$-2*{}_{Na}\Sigma[Sh(C_l)\_GUnfix * Sh(C_l)\_R)/Sh(C_l)\_R] + 2* \mathbf{Sh\_R_A}*{}_{Na}\Sigma[Sh(C_l)\_R^2/Sh(C_l)\_R] = 0$$

The 1st equation for variable **Sh_R_A** can be simplified to the folloowing form:

[6] These constraints are some analogues of the R.Hill constraints for PPPs – see equations (4)-(6) in Hill (2010). Sh_R_A, Sh_R_B, Sh_R_C – analogues of the parameters α, β, γ in R. Hill approach.

- $_{Na}\Sigma Sh(A_j)\_GUnfix + \mathbf{Sh\_R_A} * {}_{Na}\Sigma Sh(A_j)\_R = 0$

Taking into account that $_{Na}\Sigma Sh(A_j)\_R = 1$ (sum of country's shares within Regional comparison is 1), variable $\mathbf{Sh\_R_A}$ should have the following value:

$$\mathbf{Sh\_R_A} = {}_{Na}\Sigma Sh(A_j)\_GUnfix$$

The same for the variables $\mathbf{Sh\_R_B}$ and $\mathbf{Sh\_R_C}$

$$\mathbf{Sh\_R_B} = {}_{Nb}\Sigma Sh(B_k)\_GUnfix$$

$$\mathbf{Sh\_R_C} = {}_{Nc}\Sigma Sh(C_l)\_GUnfix$$

Taking into account that $\Sigma Sh(A_j)\_GUnfix$ is $\mathbf{Sh\_GUnFix\_R_A}$, etc., we have the final relations:

$$\mathbf{Sh\_R_A} = \mathbf{Sh\_GUnFix\_R_A}$$

$$\mathbf{Sh\_R_B} = \mathbf{Sh\_GUnFix\_R_B}$$

$$\mathbf{Sh\_R_C} = \mathbf{Sh\_GUnFix\_R_C}$$

It means if we want to minimize the square of deviations between country's Volume shares (unfixed and fixed World results) then the Regional volumes obtained during the global aggregation should be redistributed in accordance with the country's shares obtained in the Regional comparisons.

What procedure (LSQ or LAD) is preferable? The absolute deviations (LAD procedure) are more natural for the shares than the squares of deviations. Additionally, these are more understandable and transparent for users. However LSQ procedure is simpler technically and, most important, that this procedure leads to the same results as present CAR-Volume and CAR-PPP approaches.

The LSQ procedure for Volumes is nothing more than the **fixity of volumes ratios** which was used earlier in the ICP when the G-K method was used for the aggregation – the Regional volumes obtained during the global aggregation are redistributed in accordance with the shares obtained in the Regional comparisons [see Heston (1986) and Dikhanov (2007)])[7].

The main aim of the ICP is the GDP Volume comparison. Therefore the fixity of the Volume indicators is preferable and this approach should be used for the ICP 2011[8].

[7] This procedure leads to the use of weighted harmonic scalars for the PPPs (ICP Manual, Ch.15).

[8] This is valid, first of all, for the aggregated categories. However fixity of PPPs can be preferable at the BH level because expenditure data is usually (very) weak at the BH level.

References

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