Panel Income Changes and Changing Relative Income Inequality*

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Abstract

When economic growth (or economic decline) takes place, who benefits and who is hurt how much? The more traditional way of answering this question is to compare two or more comparable cross sections and gauge changing income inequality among countries or individuals. A newer way is to utilize data on a panel of countries or a panel of people and assess the pattern of panel income changes. How do these two approaches relate to one another? This paper shows, first, that it is possible to have all four combinations of rising or falling inequality along with divergent or convergent panel income changes, and second, under what conditions, for various measures of rising/falling inequality and various measures of divergent/convergent income changes, each of the four possible combinations can arise.
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1 Introduction

Who benefits and who is hurt how much when an economy grows or contracts? The more traditional way of answering this question is to compare data from two or more anonymous cross sections and gauge changing income inequality among individuals or households. Calculations of cross-sectional inequality measures such as Gini coefficients, income shares of particular quantiles of the income distribution, and comparisons of Lorenz curves have a long and distinguished history. A more recent technique within the anonymous tradition is to calculate Growth Incidence Curves (GICs) which, by design, compare the growth of incomes among anonymous quantiles (Ravallion and Chen, 2003).

A newer way of gauging who benefits and who is hurt is to utilize data on a panel of people and assess the pattern of panel income changes, allowing people to change quantiles. Often called income mobility analysis, the assessment of panel income changes usually is carried out by means of regressions capturing income dynamics (e.g. Atkinson *et al.*, 1992), or by constructing what are called mobility profiles (e.g. Grimm, 2007; Van Kerm, 2009) or, synonymously, non-anonymous Growth Incidence Curves (Bourguignon, 2011).

The fundamental difference between these two approaches is that the income inequality approach treats people anonymously, while the panel data approach works with the income changes of identified people in a panel, treating them non-anonymously across periods. More specifically, when looking at income inequality using such familiar tools as Lorenz curves and inequality indices, the analyst looks at the income of whoever is in the p'th position in each distribution (initial and final) regardless of whether that is the same person in one distribution as in the other. By contrast, when looking at panel income changes, the analyst first identifies which individual is in the p'th position in the initial distribution and follows that person over time, even if that person is in a different position later on.

Thus, a statement about the persons in a particular group g say, the richest 1% or poorest 10%, means different things in the two approaches. The standard inequality analysis permits statements of the type "the *anonymous* richest 1% got richer while the *anonymous* poorest 10% got poorer" while the panel data analysis makes a different type of statement: "those who *started* in the richest 1% experienced income changes of such and such amount while those who *started* in the poorest 10% experienced income changes of a different amount." To the extent that people move around within the income distribution, the two approaches provide different information.

In the literature, the anonymous approach has been explored in much more detail than the panel one. However, as Bourguignon (2011) argues in the context of income growth of countries in the world distribution of mean incomes "*if one is interested in whether global growth has been pro-poor* [...] *there does not seem to be any good reason for ignoring what happened to countries that grew fast enough to move out of the bottom deciles*" [emphasis in the original].

In practice, both approaches are meaningful. On the one hand, the panel approach is intrinsically richer, as not only can we explore changes in the (anonymous) marginal distributions across periods, but also individual transitions across them. However, for many applications the information provided by the anonymous approach is sufficient. For instance, in the debate on the political economy implications of growing inequality (Stiglitz, 2013, 2015; Bourguignon, 2015), what matters is the gap between the top and bottom of the income distribution, and not so much the income-origin of the rich and the powerful.

Numerous empirical studies have shown that the exact same data can produce markedly different patterns depending on whether the anonymous or panel approach is used. For analyses comparing the two approaches applied to the income growth of the same individuals or households, see Dragoset and Fields (2008) on the United States, Grimm (2007) on Indonesia and Peru, Khor and Pencavel (2010) on China, Palmisano and Peragine (2014) on Italy, Fields *et al.* (2015) on Argentina, Mexico, and Venezuela, and Jenkins and Van Kerm (2011) on Britain, among others. See also Bourguignon (2011) on growth of mean incomes for countries in different deciles of the world per capita income distribution.

Are cross-sectional changes favoring the anonymous rich over the anonymous poor necessarily accompanied by panel income changes favoring the panel rich over the panel poor, and likewise for the anonymous poor and panel poor?

The idea that a pattern of panel changes whereby those at the bottom gain more than those at the top necessarily results in falling inequality was first raised by Francis Galton in 1886. Later scholars demonstrated that no such implication holds, and Galton's assertion has come to be dubbed "Galton's fallacy" (see, for example, Bliss, 1999).

The literature also offers a claim regarding the opposite set of circumstances. Consider a panel of countries with per capita incomes in comparable currency units - Purchasing Power Parity-adjusted dollars, for example. Define β-divergence (convergence) as arising when a regression of final log-income on initial log-income produces a regression coefficient greater than (less than) one. Define σ-convergence (divergence) as arising when the variance of log-incomes falls (rises) from the initial year to the final year. It is proven in the literature that β-divergence measured in this way and σ-convergence measured in this way cannot arise simultaneously - more specifically, σ-convergence implies β-convergence, but β-convergence does not imply σ-convergence (Furceri, 2005; Wodon and Yitzhaki, 2006).

Is it possible to have convergent panel income changes- that is, the income changes we see following named individuals over time are decreasing in initial income- and simultaneously to have rising income inequality? Is it possible to have divergent panel income changes along with falling income inequality? Are the possibilities in times of economic growth different from those in times of economic decline? When do these different possibilities arise?

The first purpose of this paper is to derive what is possible and what is impossible. Contrary to the suggestions in the preceding paragraphs, we show that it is indeed possible to have rising or falling inequality along with convergent or divergent panel income changes, both in times of economic growth and in times of economic decline; see Table 1 and Section 2.3.

The second purpose of this paper is to derive conditions under which, for various measures of rising/falling inequality and various measures of convergent/divergent panel income changes, each of the four possibilities can arise. A number of propositions are derived; see Section 3.

Our paper is not the first one to derive conditions relating changes in relative inequality to convergence and divergence in panel income changes from a theoretical perspective. In addition to the aforementioned contributions by Furceri (2005) and Wodon and Yitzhaki (2006); Jenkins and Van Kerm (2006) decompose changes in Generalized Gini indices (Donaldson and Weymark, 1980) into two components reflecting share convergence and a term reflecting re-ranking. Similarly, Nissanov and Silber (2009) propose an alternative reconciliation of β- and σ-convergece, as defined above.

Our contribution to this literature is that unlike the studies just cited, our reconciliation of changes in inequality and panel income changes is made using very general and widely used measures of both phenomena. In particular, our analysis of inequality changes is made using three different approaches. First we look at changes in commonly used inequality indices like the coefficient of variation, the variance of log-incomes, and the Gini. Then, we provide results for the cases of Lorenz-curve dominance. Finally, in cases when the Lorenz curves cross, we also analyze changes in inequality using the family of Transfer-sensitive inequality indices, whenever one distribution third-order stochastically dominantes another one. Similarly, for the analysis of panel income changes we rely on the analysis of linear regressions between initial and final incomes, as traditionally used in studies of intra-generational income mobility (e.g. Atkinson *et al.*, 1992), inter-generational mobility (e.g. Solon, 1999), and the macro literature on absolute convergence (e.g. Barro, 1991; Sala-i-Martin, 1996).[1] By offering a reconciliation of widely used measures of inequality and panel income change, we then provide a framework that can be used by these several literatures.

Overall, the results in this paper reaffirm what has been known in the literature for some time: Whether income inequality rises or falls in the cross section is one thing. Whether panel income changes are divergent or convergent is another thing. Rising/falling inequality and divergent/convergent panel income changes are both interesting; they are, however, different.

But the results here are not just a reaffirmation. This paper goes beyond the previous literature in deriving precise conditions under which i) income inequality rises or falls, ii) panel income changes are divergent or convergent, iii) the four possibilities in Table 1 can arise, and iv) certain combinations cannot arise for particular measures of changing inequality and convergence/divergence. These conditions are derived in Section 3 and summarized in Section 4.

2 Measurement Issues and a Matrix of Possibilities

We begin by defining our terms precisely. The two key variables in this research are income inequality and panel income changes. "Income" is the term used for the economic variable of interest, which could be total income, labor earnings, consumption, or something else. The income recipient will be called a "person", but the results apply equally to households, workers, per capitas, or adult equivalents.

[1]In the macroeconomics literature the term "absolute convergence" is used when the only explanatory variable in the regression is initial income.

2.1 Income Inequality

When is income inequality rising or falling? The way we measure inequality change is completely standard (e.g. Sen, 1997; Cowell, 2011), namely, we use the Lorenz functional or a suitable inequality index to represent the inequality at two points in time and then to compare them.

Income inequality and the change in income inequality are conceptualized and measured in a number of ways. “Relative inequality” is concerned with income comparisons measured in terms of ratios, “absolute inequality” with income comparisons measured in terms of dollar differences.

A powerful and widely-used criterion for determining which of two income distributions is relatively more equal than another is the three-part Lorenz criterion, which states i) if Lorenz curve A lies somewhere above and never below Lorenz curve B, A is more equal than B, ii) if Lorenz curves A and B coincide, then A and B are equally unequal, and iii) if the Lorenz curves of A and B cross, the relative inequalities of A and B cannot be compared using the Lorenz criterion alone. Judging a Lorenz-dominant distribution to be more equal than a Lorenz-dominated one is equivalent to making inequality comparisons on the basis of four commonly-accepted relative inequality axioms: anonymity, scale-independence, population-independence, and the transfer principle (Fields and Fei, 1978).

Yet, despite its appeal, the Lorenz criterion is not universally used for two reasons: it is ordinal, and it is incomplete. When the Lorenz criterion does render a verdict about which of two income distributions is more equal than another, it can only say that A is more equal than B but not how much more equal A is than B. And when Lorenz curves cross, the Lorenz criterion cannot render a verdict.

Those analysts who seek a complete cardinal comparison of the inequalities of two income distributions are led to use one or more inequality indices. For present purposes, these indices can be put into three categories:

1. Lorenz-consistent relative inequality indices: An inequality index is Lorenz-consistent if, when one Lorenz curve dominates another, the index registers the dominant distribution as (strictly) more equal (strong Lorenz-consistency) or equally unequal (weak Lorenz-consistency). A partial listing of strong Lorenz-consistent relative inequality indices includes the Gini coefficient, Atkinson index, Theil index, and the coefficient of variation and its square. Included among the weakly Lorenz-consistent inequality indices are the income share of the richest X%,

income share of the poorest Y%, and the decile ratios (e.g. 90-10). For details, see Sen (1997) and Cowell (2011).

2. Lorenz-inconsistent relative inequality indices: An inequality index is Lorenz-inconsistent if, when one Lorenz curve dominates another, it is ever the case that the index shows the Lorenz-dominant distribution to be less equal. One commonly-used relative inequality index is Lorenz-inconsistent: the variance of the logarithms of income. This index violates the transfer principle - that is, it is possible to make a rank-preserving transfer of income from a relatively rich person to a relative poorer person and yet the index can register an increase in relative inequality (Foster and Ok, 1999; Cowell, 2011).

3. Transfer-sensitive inequality indices: These indices are Lorenz-consistent, but they can also unanimously rank distributions even in the presence of crossings in Lorenz-curves, as long as one distribution third-order stochastically dominates another. All members of the Atkinson's index family, the Theil index, and more generally all the Generalized entropy measures with parameter smaller than 2 are "transfer-sensitive". The Gini index, however is not.

In our work below, we emphasize Lorenz curve comparisons and Lorenz-consistent inequality indices. However, we give attention to the variance of log-incomes despite its Lorenz-inconsistency, because of its widespread use in the literature.

2.2 Divergent and Convergent Panel Income Changes

By definition, income mobility analysis entails looking at the joint distribution of incomes at two or more points in time. This is an analysis of panel income changes since we follow a particular individual over time. Our analysis in this paper is limited to income changes between an initial period and a final period.

The income mobility literature distinguishes six mobility concepts: time-independence, positional movement, share movement, directional income movement, non-directional income movement, and mobility as an equalizer of longer-term incomes relative to initial (Fields, 2008). For purposes of characterizing the pattern of panel income changes in this paper, the relevant concept is directional income movement among panel people - that is, who gains or loses how much, from an initial date to a final one.

Panel income changes are said to be divergent when the income recipients who started ahead on average get ahead faster than those who started behind. It is convergent when those who started ahead on average get ahead more slowly than those who started behind. It is neutral when neither is the case.

What it means to get ahead at a faster, slower, or same rate itself requires careful specification. In the macroeconomics literature, the object of interest is nearly always the growth rate in percentages, often approximated by changes in log-income (see, for example, Barro, 1991; Sala-i-Martin, 1996). On the other hand, the literature on panel income changes among individuals or households presents a more varied picture; some studies use income changes in dollars, while others use changes in log-dollars, exact percentage changes, changes in income shares, or changes in income quantiles such as deciles or centiles (see, for instance, Jäntti and Jenkins, 2015).

Much of the literature assesses divergence or convergence by assuming a linear relationship between final income and initial income or between income change and initial income. In this paper, we follow this approach as well.

Accordingly, we gauge divergence or convergence as follows. Consider a generic income variable y, which can be measured in dollars or by a strictly monotonically increasing function of dollars. We can have the levels-on-levels regression $y_1 = \alpha_y + \beta_y y_0 + u_y$ or the change-on-initial regression $\Delta y \equiv y_1 - y_0 = \gamma_y + \delta_y y_0 + u_y$. The two regressions are linked by the relationship $\delta_y = \beta_y - 1$. Divergence is said to arise when $\beta_y > 1$, or equivalently, when $\delta_y > 0$. Likewise, we have convergence when $\beta_y < 1 \iff \delta_y < 0$.

One of the monotonically increasing transformation of income that we analyze is the income shares s. Since relative inequality is concerned with the distribution of income shares, it is natural to compare it to a regression also expressed in shares. Hence, whenever the regression $s_1 = \alpha_s + \beta_s s_0 + u_s$ leads to a $\beta_s < 1$ we will denote it as there being "share convergence". An alternative way to estimate convergence in shares is through the regression of share changes in (normalized) initial ranks r_0

$$\Delta s = \kappa + \lambda r_0 + e.$$

In this case, whenever $\lambda < 0$ we will say that there is "share-on-ranks convergence". This regression will be useful in the context of analyzing convergence and changes in the Gini index.

In spite of this natural connection between relative inequality and a share-change regression, often when someone is interested in finding out whether "the rich got richer and the poor, poorer" the reference is to changes in dollars and not merely in shares. For this reason we will also study changes in dollars as well.

Finally, we may be interested in divergence or convergence of proportional changes. In many applications economists have been interested in studying whether proportional income changes are convergent or divergent. In particular they have studied whether on average initially richer individuals had proportional income changes larger than those of initially poorer individuals.

We can approximate proportional changes using a log-log regression or we can measure them exactly, in which case we would want to regress the (exact) proportional change in dollars on initial dollars: $\text{pch d} \equiv (d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}$. In the latter case, exact proportional changes are divergent or convergent according to whether θ is greater or less than zero.

Convergence in dollars and divergence in proportional changes cannot coexist in periods of economic growth, since if the initially poor gain more in dollars than the initially rich (i.e., there is convergence in dollars) then proportional changes are necessarily convergent as well. However, the same is not true in periods of economic decline. To appreciate this, consider a hypothetical two-person economy with the following income transition

$$[2, 50] \rightarrow [1, 45]$$

where the poor individual lost 1 dollar while the rich one lost 5 dollars. By our definition there is convergence in dollars. Yet the 1-dollar loss represented half of the poor individual's income, while the 5-dollar loss represented only a 10% loss for the rich individual. Hence, in this example there was convergence in dollars but divergence in proportional changes. The previous example illustrates why an analysis of proportional income changes is an important complement to the analysis of dollar changes and share changes, especially during times of economic decline.

In summary, in our paper we will focus our attention on panel changes of income measured in dollars and shares, as well as on proportional dollar changes, both measured exactly or approximated through logarithms.

2.3 A Matrix of Possibilities

We have identified three ways of determining the direction of change in relative inequality - i) Lorenz-improvement and Lorenz-worsening, ii) Change in a Lorenz-consistent relative inequality index, and iii) Change in Lorenz-inconsistent relative inequality measures - and four ways of assessing divergence or convergence: i) Dollar changes, ii) Share changes, iii) Log-dollar changes, and iv) Exact Proportional changes.

Can each possible combination of rising or falling relative inequality and divergent or convergent panel income changes arise? We show in this paper that the answer is yes, provided they are measured suitably. Table 2 displays examples for each of the possible combinations.

To demonstrate the possibilities of most of the combinations, just two people are needed. But to get the remaining combinations, we need to complicate the examples by adding more people and choosing our measures carefully.

This matrix of possibilities shows a number of other things:

- Many but not all possibilities involve a Lorenz-dominance relationship (27 out of 32 examples to be precise). When such a result holds, it is stronger than a result for a particular inequality index, because it holds for *all* Lorenz-consistent inequality indices.
- All four convergence rows are consistent with Lorenz-worsening and Lorenz-improvement, both in times of positive economic growth and in times of negative economic growth.
- Falling relative inequality as gauged by Lorenz-improvement is consistent with all four types of convergence but not with divergent share changes, divergent proportional changes, or divergent dollar changes in times of negative economic growth.
- Falling relative inequality as gauged by a suitably chosen Lorenz-consistent inequality measure is consistent with all types of divergence. However, some of these combinations can only arise when the two Lorenz curves cross.
- Even in the absence of positional change, it is possible to have Lorenz-worsening together with convergent dollar changes in periods of eco-

nomic decline, and with convergent log-dollar changes, both under positive and negative growth.[2]

Examples prove possibilities; they do not produce exact conditions. In the next section, we derive a number of necessary and sufficient conditions for the various possibilities.

3 Mathematical Results

In this section we analytically develop a set of results that establish the connection between changes in relative inequality and our several income change concepts. First, we present some notation and definitions, then in sections 3.2-3.5 we derive conditions on possibilities and impossibilities for different ways of measuring inequality and income changes. In everything that follows we consider regressions done on population and abstract from all issues of inference.[3]

3.1 Notation and Definitions

Consider an economy with n individuals observed over two time periods, initial (or 0), and final (or 1).

Denote by d_{it} the income of individual i in period t measured in constant monetary units (e.g., real dollars). We drop the individual subindex i to denote vectors, e.g., $d_t = (d_{1t}, d_{2t}, \ldots, d_{nt})'$.

The basic building block of panel data analysis is the panel data matrix $\mathbf{D} = [d_0, d_1]$. Divide each column of $\mathbf{D}$ by its respective mean, i.e. divide the dollar incomes by mean income in that year μ_t. The resulting matrix of shares can be written as $\mathbf{S} = [s_0, s_1]$. In addition to income shares, we will also deal with other strict monotonic transformations of income, like log-incomes, denoted by $\ln d_t$.

More generically, when a result can be derived both for income in dollars and for a transformation of it, we will denote by $y_t = f(d_t)$ the income variable transformed by the strictly monotonically increasing function $f(\cdot)$.

[2]It is easy to generate examples in these cells using only two individuals who change positions between periods.

[3]Another assumption we make is that the initial distribution of income has non-zero dispersion, e.g. $V(y) \neq 0$, as otherwise our regression coefficients would not be defined.

A crucial feature of the panel data matrix $\mathbf{D}$ is that it involves pairs of incomes for each individual, which implies that if the i-th element of d_0 is moved, the i-th element of d_1 must also be moved to the same row. In other words, in panel data analyses we are allowed to permute entire rows of $\mathbf{D}$, a property called *Multi-period Anonymity.* This contrasts with the property of *Single-period Anonymity* (or simply *Anonymity*) commonly used in the analysis of cross-sectional inequality. In the latter case, we are allowed to separately permute a given column of $\mathbf{D}$ without necessarily permuting the elements in other columns of the data matrix. In mobility studies then, the assumption of single-period anonymity is replaced by multi-period anonymity, where the income *trajectories* matter without having to look at the names of the particular individuals experiencing such trajectories.

For the most part, income vectors and their transformations are sorted in ascending order of individuals' *initial-period* incomes.[4] An exception to this is the final income-share vector s_c, where the sorting is ascending in *final-period* income; such sorting is important for Lorenz curve calculations.

Definition 1 ***Vector of Final Shares in Ascending Order.***
Let $P(\cdot)$ be a permutation operator. Then, define $s_c = (s_{1c}, \ldots, s_{nc})$ as the counterfactual final income-share vector when final incomes are sorted in ascending order of final *income, i.e.*

$$s_c \equiv P(s_1) \quad \text{such that} \quad s_{ic} \leq s_{jc} \;\; \forall\, i \leq j. \tag{1}$$

It is useful to illustrate the relationship between $\mathbf{D}$, $\mathbf{S}$, and $\mathbf{s_c}$ with a simple example. In particular, we display below a particular panel data matrix $\mathbf{D}$, together with its corresponding matrices $\mathbf{S}$, and $\mathbf{s_c}$.

$$\mathbf{D} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 10 & 9 \end{bmatrix}; \; \mathbf{S} = \begin{bmatrix} 0.23 & 0.69 \\ 0.46 & 0.23 \\ 2.31 & 2.08 \end{bmatrix}; \; \mathbf{s_c} = \begin{bmatrix} 0.23 \\ 0.69 \\ 2.08 \end{bmatrix}$$

Another occasion where we don't sort vectors in ascending order of initial incomes is r_{it}, which denotes the population-normalized rank of individual i in period t, when the distribution in period t is sorted in ascending order of income of *that same period.* In other words, if R_{it} is the rank of individual i when the distribution is sorted in ascending order of income in period t, the normalized rank equals $r_{it} = R_{it}/n$.

With this notation we can now define the Lorenz Dominance criterion.

[4]This sorting is immaterial for the convergence regressions.

Definition 2 ***Lorenz Dominance.***
Let s_{j0} be the initial income-share of the individual in position j, when shares are sorted in ascending order of initial income. Let s_{jc} be the final income-share of the individual in position j, when shares are sorted in ascending order of final income. The final income distribution Lorenz-dominates the initial one whenever

$$\begin{aligned} &s_{1c} + s_{2c} + \ldots + s_{jc} \geq s_{10} + s_{20} + \ldots + s_{j0} \quad \text{for j = 1, 2, \ldots, n - 1 and} \\ &s_{1c} + s_{2c} + \ldots + s_{jc} > s_{10} + s_{20} + \ldots + s_{j0} \quad \text{for some j < n.} \end{aligned} \tag{2}$$

As previously mentioned, having the final period distribution Lorenz-dominate the initial one means that the final distribution is more equally distributed than the initial one according to this criterion. This situation is sometimes also referred as a "Lorenz-improvement" when going from d_0 to d_1. Similarly, if the previous inequalities are reversed we talk of a "Lorenz-worsening".[5]

Following standard notation, we will denote the Lorenz Curve of income in period t by LC_t, and $LC_1 \succ LC_0$ means that the Lorenz curve in period 1 dominates that of period 0, namely incomes in period 1 are more equally distributed than the ones in period 0 according to the Lorenz-criterion. If the domination is weak we denote it as $LC_1 \succeq LC_0$, which means that incomes in period 1 are at least as equally distributed as those in period 0 by the Lorenz criterion. $I(\cdot)$ will be used to denote an arbitrary relative inequality measure.

Another concept that we will need throughout the paper is that of a Rank-Preserving Transfer, defined next.

Definition 3 ***Equalizing Rank-Preserving Transfer.***
A rank-preserving equalizing transfer $h > 0$ is a transfer of income between

[5]The literature usually expresses condition (2) using income as a share of total income. In order to make an easier link with the regressions involving share changes we express it in terms of shares of mean income. It is obvious that the Lorenz curves are the same in the two cases, and hence the inequality comparisons using the Lorenz criteria are also the same.

two individuals with ranks i and j for $d_{j0} > d_{i0}$, such that:

$$\begin{array}{ll} d_{k0} = d_{k1} & \text{for } k \neq i, j, \\ d_{j1} = d_{j0} - h, & \\ d_{i1} = d_{i0} + h, & \text{where:} \\ \text{if } j = i + 1, & h < (d_{j0} - d_{i0})/2; \\ \text{if } j > i + 1, & h < \min[(d_{i+1,0} - d_{i0}), (d_{j0} - d_{j-1,0})]. \end{array}$$

A rank-preserving disequalizing transfer is defined similarly.[6] Equalizing transfers are sometimes called "progressive transfers", while disequalizing transfers are sometimes called "regressive transfers".

Finally, recall the definitions of divergence and convergence and the accompanying notation.

Definition 4 ***Convergence and Divergence***
For a generic income variable y, define the levels-on-levels regression

$$y_1 = \alpha_y + \beta_y y_0 + u_y \tag{3.1}$$

or the change-on-initial regression

$$\Delta y \equiv y_1 - y_0 = \gamma_y + \delta_y y_0 + u_y. \tag{3.2}$$

Divergence arises when $\beta_y > 1$, or equivalently, when $\delta_y > 0$. Convergence arises when $\beta_y < 1$, or equivalently, when $\delta_y < 0$. Otherwise, the income change patterns are deemed neutral.

An alternative way of estimating convergence in shares is through the share-change on initial-rank regressions

$$\Delta s = \kappa + \lambda r_0 + e, \tag{4}$$

in which case there will be "share-on-ranks" convergence whenever $\lambda < 0$, divergence if $\lambda > 0$, otherwise the share changes are deemed neutral.

Finally, define the regression of the exact proportional changes in dollars on initial dollars

$$\text{pch d} \equiv (d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}. \tag{5}$$

Divergence arises when $\theta > 0$. Convergence arises when $\theta < 0$. Otherwise, the exact proportional change patterns are deemed neutral.

[6] In this case, the final income of the poorer individual will be $d_{i1} = d_{i0} - h$, the final income of the richer individual will be $d_{j1} = d_{j0} + h$, and the last two conditions are replaced by $h < \min[d_{i,0} - d_{i-1,0}, d_{j+1,0} - d_{j,0}]$.

If the income variable is expressed in dollars, we subscript the parameters of the generic regressions (3.1) and (3.2) with "d". If the income variable is expressed in log-dollars, we subscript the parameters with "log", and if the income variable is expressed in shares, we subscript the parameters with "s".

One final concept that we will need in order to establish some of the results in the paper is "quadrant-dependence" (Lehmann, 1966), defined next.

Definition 5 ***Quadrant Dependence***
Let X and Y be random variables with marginal cdf's $F_x(x)$ and $F_y(y)$, respectively, and with joint bivariate cdf $H(x, y)$. We say the pair (X, Y) is

i) positively quadrant-dependent if $H(x, y) \geq F_x(x)F_y(y) \quad \forall x, y$,

ii) negatively quadrant-dependent if $H(x, y) \leq F_x(x)F_y(y) \quad \forall x, y$.

More generally, when (X, Y) satisfy either i) or ii) we say the pair is quadrant-dependent.

The concept of quadrant-dependence is an alternative to the more commonly-used covariance for measuring association between two random variables. In this case, dependence is evaluated by comparing the probability of any quadrant $(X \leq x, Y \leq y)$ with the probability that would occur in case of independence between the two variables. In particular, condition i) (ii)) is a positive(negative) concept of dependence because (X, Y) are more(less) likely to jointly be large or small than they would be in the case of independence.

It can be easily shown that quadrant-dependence is a stronger concept than a linear association, as measured by a covariance (e.g. Lemma 3 in Lehmann, 1966).[7] In our context, the existence of quadrant-dependence will prove useful because it allows us to derive results when one of our income variables is transformed by a monotonic function.

In what follows we will derive our results under two implicit assumptions. First, we will assume that between the initial and final periods, the income in dollars of at least one individual changes. In other words, we won't occupy ourselves with the case where $\Delta d_i = 0$ for all individuals. In this case of perfect immobility, inequality will remain unchanged, irrespectively of how it is measured. Also, the slope coefficients for all of our income change regressions (e.g. δ_y, λ, θ) will be exactly zero.[8]

[7] Positive(negative) quadrant-dependence between two variables implies a positive(negative) covariance between them, but not the other way around.

[8] In the case of the levels-on-levels regression, the slope, β_y, will equal 1.

The second assumption we will maintain is that in the initial period, the income distribution is not completely equal. That is, we assume $V(y_{i0}) > 0$. If by contrast, we were to allow cases where $V(y_{i0}) = 0$, all the slope coefficients in our regressions would be undefined. Having excluded those two extreme cases we can now proceed to present our results. The proofs of all the results are included at the end of the paper.

3.2 Inequality Measures and Panel Income Changes

We begin our analysis by presenting results that link our four income change regressions with some commonly used inequality measures. It is important to emphasize that Lorenz-domination or the absence thereof is not required for any of the results in this section. Results under Lorenz-dominance will follow in section 3.3.

3.2.1 Variance-based Measures

We begin with measures that are related to variance conditions. The reason to begin with this family of inequality indices is that variances can naturally be related to regression coefficients like the ones defined above.

Variance-based inequality measures are widely used in the literature. Not only is the variance a natural measure of dispersion, but in the macro and labor literatures, it is common to assess changes in relative inequality by focusing on the variance of log-incomes, in spite of its Lorenz-inconsistency, as already noted. Furthermore, it can easily be shown that the variance of shares is the square of the coefficient of variation, a Lorenz-consistent inequality measure (see Lemma 1 below).

Our first result links the variance of any monotonically increasing function of income in dollars $y = f(d)$ (e.g. logarithms, shares, etc.) and the coefficient of a regression of the changes in this generic variable y on its initial level y_0. Namely, we present a result concerning the relationship between the changes in the variance of y and the coefficient δ_y in regression (3.2).

Proposition 1 ***Convergence and Changes in Variance for the Class of Monotonic Transformations of Income in Dollars.***
Let $f(d)$ be any monotonically increasing function of income in dollars and denote the value of this function by y. Consider the change-on-initial regression (3.2). Then:

i) If $\Delta V(y) < 0$, then $\delta_y < 0$, i.e., the changes in generic income are convergent.

ii) If $\Delta V(y) > 0$, the changes in y can be either convergent or divergent.

It is useful to also present the first result in its contrapositive mode. We do this in the next corollary.

Corollary 1 *If $\delta_y \geq 0$, then $\Delta V(y) \geq 0$.*

Proposition 1 and its Corollary show that non-convergence in the changes of a monotonically increasing function of dollars $y = f(d)$ implies a rising variance of this function, (or alternatively a non-rising variance of y implies convergent changes of y). However, convergence does not imply a falling variance: $\delta_y < 0 \nRightarrow \Delta V(y) < 0$.

These results pertain to any monotonically increasing function of income, as long as we use the <u>same</u> function $y = f(d)$ as dependent and independent variables, i.e. as long as we run share-changes on initial shares, dollar changes on initial dollars, etc. As previously mentioned, a particular case of this result for the variance of logs and the coefficient in a log-change regression, was derived independently by Furceri (2005) and Wodon and Yitzhaki (2006).

Applied to our selected regressions, these results relate convergence coefficients to three inequality measures (variance of dollars, variance of log-dollars, and variance of shares). However, only one out of these three measures, the variance of shares, is Lorenz-consistent, a result that follows immediately from the following Lemma and the fact that the coefficient of variation is (strongly) Lorenz-consistent.

Lemma 1 ***Variance of Shares and Coefficient of Variation***
Let $CV(d)$ denote the coefficient of variation of income, then

$$CV^2(d) = V(s).$$

In other words, Proposition 1 and Lemma 1 together give us a link between a share-change regression

$$\Delta s \equiv s_1 - s_0 = \gamma_s + \delta_s s_0 + u_s. \tag{6}$$

and the coefficient of variation. In addition to that, we can derive a result relating dollar-change regressions

$$\Delta d = \gamma_d + \delta_d d_0 + u_d \tag{7}$$

to the coefficient of variation as well.

Proposition 2 ***Convergence in Dollars, Changes in the Coefficient of Variation, and Economic Growth.***

Let β_d be defined by the final-on-initial regression (3.1) when income is measured in dollars, and denote the correlation coefficient from this regression by ρ_d. Let $CV(d_t)$ denote the coefficient of variation of income at period t, and let g denote the economy-wide growth rate in incomes between year 0 and year 1. Then there is divergence/convergence in dollars as follows:

$$\beta_d \gtreqless 1 \qquad (\text{i.e. } \delta_d \gtreqless 0) \iff \rho_d \frac{CV(d_1)}{CV(d_0)}(1+g) \gtreqless 1. \tag{8}$$

In fact, we can go further and provide precise conditions for when we can observe the four combinations of falling/rising inequality, as measured by variance-based measures of inequality, together with convergent/divergent income changes as measured by the previous regressions. These conditions are presented in Table 3 for the generic income variable y, and in Table 4 for the coefficient of variation of incomes in dollars paired with a dollar-change regression.

It is instructive to illustrate how these conditions operate in a simple two-person example. Consider in particular an economy in which the anonymous distribution of income in dollars changes from (1,3) to (1,5). The underlying possibilities are:

$$\begin{aligned} \text{Case I}: &\quad [1,3] \rightarrow [1,5] \\ \text{Case II}: &\quad [1,3] \rightarrow [5,1]. \end{aligned}$$

In Case I, dollar changes are divergent, and the variance of dollars rises, so we are in the cell (2,2) of Table 3. In Case II, however, the increase in the variance of dollars $\Delta V(y)$ equals 6, while the variance of dollar changes, $V(\Delta d)$ equals 18. This puts us in cell (1,2), where convergent dollar changes co-exist with a rising variance of dollars. In other words, the condition in cell (1,2) illustrates that if the dispersion income changes is large enough, then it is possible to reconcile rising inequality (as measured by the variance of y), together with convergent income changes.[9]

[9] A similar argument can be made for share and log-dollar changes when paired with their respective variances.

A look at the two panels of Table 4 shows that to make a rising coefficient of variation compatible with convergent dollar changes, we must either have a sufficiently strong economic decline ($g < 0$) or a sufficiently low ρ_d.

Consider an economy in which economic growth has taken place (i.e., $g > 0$) and income inequality as measured by the coefficient of variation has risen ($CV(d_1) > CV(d_0)$). If initial and final incomes were perfectly positively correlated - that is, if ρ_d were equal to +1- then applying equation (8) we would know that panel income changes in dollars would necessarily be divergent (i.e., $\beta_d > 1$). However, if initial and final incomes are positively correlated but not perfectly positively correlated (i.e., $0 < \rho_d < 1$), room is left open for the possibility that a growing economy with rising income inequality might also have convergent dollar changes. Moreover, equation (8) also tells us that the smaller is ρ_d, the more room there is for positive economic growth, rising income inequality, and convergent dollar changes to coexist.

Some analysts may implicitly be supposing that income recipients who are high (low) to begin with will inevitably be high (low) at a later point in time. Whether or not this is the case is an empirical question. The answer should not, however, be assumed.

If during periods of economic decline the dollar losses of the poor are larger than those of the rich, i.e., if there is divergence in dollars, then the income share of the rich will grow and so will inequality. This accounts for the impossibility result in cell (2,1) in part B of Table 4.

What if economic growth is positive and dollar changes are divergent? In that case the dollar gains of the initially poor can be smaller than those of the initially rich, yet the share gains of the anonymous poor can be higher than the share gains of the anonymous rich (in which case there would be a fall in relative inequality as in cell (2,1) in part A of Table 4). An example from our Matrix of Possibilities (Table 2) illustrates this. Consider the transition [5, 20] $\rightarrow$ [7, 23]. In this case, $\delta_d = 0.067$, yet the CV falls by 0.067.

In more precise terms, as with any relative inequality index, the coefficient of variation is independent of the measurement scale of income; yet the coefficient of a dollar-change regression is affected by proportional dollar-changes. Hence, if positive economic growth is strong enough, it can generate divergence in dollars by proportionally increasing incomes by $1 + g$, even when relative inequality is falling.

From now on, we will present possibility/impossibility results in Propositions and Corollaries, and express in Tables the precise conditions for each combination of falling/rising inequality vs convergent/divergent income changes.[10]

To close this section on variance-based inequality measures, we present a result relating the coefficient of variation to a regression of exact proportional changes under the more stringent condition of quadrant-dependence (see Definition 5). First though, we present two useful Lemmas. The first result is useful in establishing connections between share changes and the slope of the exact proportional change regression (5). The second Lemma establishes some key properties of variables that satisfy quadrant-dependence.[11]

Lemma 2 ***Share Changes and Exact Proportional Changes***

Let θ be the slope of the exact proportional change regression (5) *then:*

$$sign(\theta) = -sign\left(E\left[\frac{s_1 - s_0}{s_0}\right]\right).$$

Lemma 3 ***Properties of Quadrant-Dependent Variables***

Let (X, Y) be a bivariate random variable. Let $f(X)$ be a strictly monotonically increasing function of X. Similarly, let $g(X)$ be a strictly monotonically decreasing function of X. If (X, Y) is quadrant-dependent, then

i) $sign(cov(X, Y)) = sign(cov(f(X), Y)$

ii) $sign(cov(X, Y)) = -sign(cov(g(X), Y)$

provided the covariances exist.

Proposition 3 ***Convergence in Exact Proportional Changes and Changes in the Coefficient of Variation***

Let θ be defined by the exact proportional change regression (5). Assume that the bivariate random variable, $(s_0, \Delta s)$, is quadrant-dependent. Then:

i) If $\Delta CV(d) < 0$, then $\theta < 0$, i.e., the exact proportional changes are convergent.

ii) If $\Delta CV(d) > 0$, the exact proportional changes can be either convergent or divergent.

[10]The derivations of the conditions in all the tables are included in an online appendix.

[11]The Lemma follows easily from the results derived in Lehmann (1966). We present a proof for the sake of completeness.

Furthermore, for part i) in this last proposition, we can provide the corresponding contrapositive form as a Corollary.

Corollary 2 *Under the assumptions stated in Proposition 3, if $\theta \geq 0$, then $\Delta CV(d) \geq 0$.*

3.2.2 Gini Index

The Gini index is probably the most widely used inequality measure in the literature. In this section we derive conditions relating our panel regressions to this index.

We begin by relating changes in the Gini and share-on-ranks convergence, i.e. share convergence, as gauged by equation (4), i.e.,

$$\Delta s = \kappa + \lambda r_0 + e.$$

Proposition 4 ***Share-on-Ranks Convergence and Changes in the Gini***

Let G_t be the Gini index in period t, and let λ be given by equation (4). Then:

i) If $\Delta G < 0$, then $\lambda < 0$, i.e., the share-on-rank changes are convergent.

ii) A rising Gini may be consistent with either convergent or divergent share-on-rank changes as measured by λ.

As before, we can also establish a set of conditions relating Gini changes to share-on-ranks convergence. We present these conditions in Table 5.

In addition to the previous results, we can also establish a connection between changes in the Gini and dollar-changes and exact proportional changes, under the assumption of quadrant-dependence. We present those results next.[12]

Proposition 5 ***Convergence in Dollars, Changes in the Gini, and Economic Growth***

Let δ_d be defined by the regression of change in dollars on initial dollars (7), and let g denote the economy-wide growth rate in incomes between year 0 and year 1. Assume that the bivariate random variable, $(r_0, \Delta s)$ is quadrant-dependent. Then:

[12] Wodon and Yitzhaki (2006) derived a different relation between changes in the Gini and a convergence coefficient using what they call a "Gini-regression" of final income on initial income.

i) If $\Delta G < 0$ and $g \leq 0$, then $\delta_d < 0$, i.e., the dollar changes are convergent.

ii) A rising Gini may be consistent with either convergent or divergent dollar changes as measured by δ_d.

Proposition 6 ***Convergence in Exact Proportional Changes and Changes in the Gini***

Let θ be defined by the exact proportional change regression (5). Assume that the bivariate variables $(r_0, \Delta s)$ are quadrant-dependent. Then:

i) If $\Delta G < 0$, then $\theta < 0$, i.e., the exact proportional changes are convergent.

ii) A rising Gini may be consistent with either convergent or divergent exact proportional changes as measured by θ.

In this section we have shown how the coefficients from our different income-change regressions relate to variance-based inequality measures (including the coefficient of variation) and to the Gini index. In some of the relations derived, we need to invoke additional conditions on the data, such as quadrant-dependence, in order for the results to hold.

We now turn to results linking our income change regressions to changes in inequality under Lorenz dominance.

3.3 Lorenz Dominance and Income Changes

In spite of the wide use of the Gini, the coefficient of variation, and the variance of logs, researchers often use other Lorenz-consistent inequality measures like the Theil, various decile ratios, among others. Not only is the Lorenz Dominance criterion the most accepted way of judging whether relative inequality has risen or fallen, but also whenever this criterion provides an ordering of the inequalities of two distributions, all Lorenz-consistent indices agree with that ordering.

It turns out that we can find a set of useful results linking Lorenz Dominance to our previous four regression methods. We present those results next. It then follows that all these results also apply to the family of Lorenz-consistent inequality indices whenever there are no crossings of Lorenz curves. In this sense, the results presented in this section are more general than the ones of section 3.2. The price to pay for the greater generality in this first sense, is however, that the results here derived will not apply in the case of Lorenz-crossings, so in this second sense, the results are less general.

3.3.1 Lorenz Dominance and Share Changes

In this section we derive a connection between the Lorenz Dominance criterion

$$\begin{array}{ll} s_{1c} + s_{2c} + \ldots + s_{jc} \geq s_{10} + s_{20} + \ldots + s_{j0} & \text{for j = 1, 2, \ldots, n - 1 and} \\ s_{1c} + s_{2c} + \ldots + s_{jc} > s_{10} + s_{20} + \ldots + s_{j0} & \text{for some j < n.} \end{array} \quad (2)$$

and a share-change regression

$$\Delta s = \gamma_s + \delta_s s_0 + u_s. \quad (6)$$

Equations (2) and (6) both involve initial and final income-shares. However, the final period shares appear sorted differently in the two expressions. More specifically, in condition (2), final shares s_c are sorted in ascending order of *final* shares, while in equation (6) final shares s_1 preserve the order of *initial* shares.

It is easy to show that the sign of the coefficient δ_s in regression (6) is determined by the sign of the covariance

$$cov(\Delta s, s_0) = \frac{\sum_i (s_{i1} - s_{i0}) s_{i0}}{n},$$

since average share changes are zero by construction.

Using vector s_c as defined in (1), we can decompose this covariance as

$$cov(\Delta s, s_0) = \frac{\sum_i [(s_{i1} - s_{ic}) + (s_{ic} - s_{i0})] s_{i0}}{n}.$$

That is, whether share changes are convergent or divergent is determined by the sum of two terms, a structural mobility term and an exchange mobility term:

$$\begin{aligned} SM &= \frac{\sum_i (s_{ic} - s_{i0}) s_{i0}}{n} \\ XM &= \frac{\sum_i (s_{i1} - s_{ic}) s_{i0}}{n}. \end{aligned} \quad (9)$$

SM captures the component of the covariance associated with changes in the shape of the income distribution for anonymous people, and XM is the component of the covariance associated with positional change, under a fixed marginal distribution.[13]

We can derive the following two key Lemmas for these terms.

Lemma 4 *Let SM be given by equation (9), then:*

i) A Lorenz-improvement ($LC_1 \succ LC_0$) *implies* $SM < 0$.

ii) A Lorenz-worsening ($LC_1 \prec LC_0$) *implies* $SM > 0$.

In other words, in cases of Lorenz-dominance, the sign of SM fully reflects whether there has been a fall or a rise in inequality judged by the Lorenz-criterion.

As previously mentioned, when looking at income changes we care not only about how the anonymous distribution of income evolves, but also about who moved to a different position across periods. This is reflected by the transition from s_c to s_1. In this transition, share changes will be convergent, since in the reranking of individuals there will always be a positive transfer of income shares from a relatively richer individual to a poorer one. This is expressed in the following Lemma.

Lemma 5

$$XM = \frac{\sum_i (s_{i1} - s_{ic}) s_{i0}}{n} \leq 0.$$

With these two results we can proceed to analyze the connection between share mobility and changes in inequality as measured by Lorenz comparisons.

Proposition 7 ***Convergence in Shares and Lorenz Dominance***

i) A Lorenz-improvement ($LC_1 \succ LC_0$) implies share convergence ($\delta_s < 0$).

ii) A Lorenz-worsening is consistent with either convergent or divergent share changes.

The intuition (and proof) behind this proposition is related to a well-known result in the inequality literature stating that an equalization in the Lorenz sense can be achieved by a series of income transfers from richer

[13] This is so because if positions were to remain unchanged, i.e. $s_c = s_1$, the entire share change would be due to a change in the shape of the distribution, $s_c - s_0$.

to poorer individuals that keep unaltered the individual ranks between the initial and the final periods (see for instance Fields and Fei, 1978).

These progressive transfers generate by construction convergent share changes in the transition from s_0 to s_c (Lemma 4). However, when going from s_0 to s_1, we also need to consider the transition from s_c to s_1. In this last step the shape of the income distribution remains unchanged and pairs of individuals swap incomes and therefore positions. As we saw in Lemma 5, this positional rearrangement leads to convergent share-changes always.

Hence, in the case of a Lorenz-improvement, both components go in the same direction, and share changes are convergent. However, in the case of a Lorenz-worsening, the two components will move in opposite directions, and depending on which force is dominant there will be convergence or divergence in shares as measured by δ_s in equation (6).

In contrast, if all individuals keep their same rank in the initial and final distributions (i.e. if there is zero positional mobility), vector s_c will equal the final share vector s_1, and the sign of δ_s is determined exclusively by SM. Given Lemma 4 and the connection between SM and δ_s, in the absence of positional changes, we have that a Lorenz-worsening will lead to divergent share changes.

In other words, as long as we restrict ourselves to the case of no positional mobility and no crossings of Lorenz curves, share mobility and changes in inequality fully align, in the sense that rising inequality as gauged by Lorenz-worsening only occurs with divergent share-changes and falling inequality as gauged by Lorenz-improvement only occurs with convergent share-changes.

When share changes are divergent, the income shares of the rich grow relative to others' shares (irrespectively of whether there is positional change or not). This should lead to disequalization. Hence, the only possible way to register a fall in inequality in this instance is for Lorenz curves to cross.[14] We express this as a corollary.

Corollary 3 *If share changes are non-convergent ($\delta_s \geq 0$) then either a weak Lorenz-worsening has taken place $LC_0 \succeq LC_1$, or the Lorenz curves of incomes in periods 0 and 1 cross.*

This corollary is just the contrapositive of Proposition 7.i), hence we omit its proof.

[14] As is well known, when Lorenz curves cross, a Lorenz-consistent measure can always be found showing rising inequality and another Lorenz-consistent measure can be found showing falling inequality.

Proposition 7 and Corollary 3 give the relation between Lorenz-dominance and share changes, while Table 6 gives the precise conditions under which each combination of convergent/divergent share changes can occur under Lorenz-dominance.

3.3.2 Lorenz Dominance and Changes in Dollars

While the previous subsection established a clear connection between change in inequality as gauged by the Lorenz criterion and share changes, as previously mentioned, on many occasions our interest is not the changes in shares but the changes in dollars.

In this section we establish a condition relating changes in inequality under Lorenz-dominance and a dollar-change regression. In order to derive such a connection, it is useful to express the dollar-change regression (7) in its final-on-initial form

$$d_1 = \alpha_d + \beta_d d_0 + u_d \tag{10}$$

and to recall that in such a case convergence will occur whenever $\beta_d < 1$ (or equivalently $\delta_d < 0$). Similarly, we can define a final-on-initial share regression

$$s_1 = \alpha_s + \beta_s s_0 + u_s. \tag{11}$$

Using these regressions we can establish the following result.

Lemma 6 *Let μ_t denote the mean income in period t, β_d and β_s denote the convergence coefficients given by regressions (10) and (11) in dollars and in shares, respectively, and g denote the economy-wide growth rate in incomes between year 0 and year 1. Then*

$$\beta_d = \beta_s \frac{\mu_1}{\mu_0} = \beta_s(1+g).$$

We can now derive a necessary condition relating dollar-changes and Lorenz Dominance.

Proposition 8 ***Convergence in Dollars and Lorenz Dominance***
Let δ_d be defined by the change-on-initial regression (7) when income is measured in dollars, and let g denote the economy-wide growth rate in incomes between year 0 and year 1.

i) If $g \leq 0$, a Lorenz-improvement ($LC_1 \succ LC_0$) implies convergence in dollars ($\delta_d < 0$).

ii) A Lorenz-worsening ($LC_1 \prec LC_0$) is compatible with both convergence and divergence in dollars.

As in previous cases, we can express the additional corollary

Corollary 4 *When economic growth is negative, if dollar changes are non-convergent ($\delta_d \geq 0$) then either a weak Lorenz-worsening has taken place $LC_0 \succeq LC_1$, or the Lorenz curves of incomes in periods 0 and 1 cross.*

Similar to Proposition 5 linking changes in the Gini index and convergence in dollar changes, we only have a relation of necessity between Lorenz-improvements and dollar changes in the case of negative growth.[15] Table 7 contains the precise conditions for each possible cell to arise under different growth scenarios. A quick comparison reveals that this table bears close resemblance to Table 4. We will not repeat the intuition here for each cell. Instead, we emphasize a new feature that arise in this context.

First, by looking at cell (1,2) in part A, we notice that in order to have convergent dollar changes and Lorenz-worsening under positive economic growth, there must be positional changes, and the exchange mobility brought forth by these changes needs to be sufficiently large, as expressed by the condition $|XM| > SM$. In contrast, the corresponding cell (1,2) in part B under economic decline imposes no condition whatsoever on positional change. In other words, not only can we have convergent dollar changes together with Lorenz-worsening, but this can happen even when individuals keep their initial position. The reason for this is that if economic decline is strong enough, the dollar losses of the initially rich can be larger than those of the rest of the population, even when the relative income share of the rich increases. For an example, refer again to the corresponding cell in our Matrix of Possibilities (Table 2).

3.3.3 Lorenz Dominance and Proportional Income Changes

In this section we explore the relationship between proportional changes in income, Lorenz-improvement/worsening, and the change in the variance of log-dollars.

[15]In spite of not having a relation of necessity between Lorenz-improvements and convergent dollar changes under all growth scenarios, we can establish such a relation if we measure absolute rather than relative inequality. In particular, we can derive a proposition similar to Proposition 7 if we use Absolute Lorenz Curves (Moyes, 1999). See online appendix for further details.

Log-Income Approximation

The most common way to measure proportional convergence is by approximating proportional changes by changes in log-income and estimating a double-log regression

$$\Delta \ln d = \gamma_{\log} + \delta_{\log} \ln d_0 + u_{\log} \tag{12}$$

or its equivalent final-on-initial form $\ln d_1 = \alpha_{\log} + \beta_{\log} \ln d_0 + u_{\log}$. Similarly a common way of determining whether inequality is increasing or decreasing is to look at the variance of log-incomes.

As we now show, doing things in these ways can be seriously misleading.

Consider the following example:

$$[1, 1, 1, 1, 1, 1, 1, 1, 6, 9] \rightarrow [1, 1, 1, 1, 1, 1, 1, 1, 7, 8].$$

The richest person (call him Bill Gates) has transferred \$1 to the next richest person (call him Carlos Slim), which is a clear Lorenz-improvement. Inequality therefore falls by the Lorenz criterion and accordingly for any Lorenz-consistent inequality measure. However, the variance of log-incomes is not Lorenz-consistent (Foster and Ok, 1999; Cowell, 2011), and it shows an increase from 0.716 to 0.721 despite the Lorenz-improvement. Moreover, a rank-preserving transfer in dollars from the richest person to anyone lower down in the income distribution should be deemed convergent, as it brings convergence in dollars (in this case $\beta_d = 0.96$). However, as this example shows, if we regress final log-dollars on initial log-dollars, we obtain $\beta_{\log} = 1.00045 > 1$, and hence find divergence in log-dollars. Thus, in this example, a Lorenz-improvement has taken place and yet the regression of final log-income on initial log-income registers divergence and the variance of log-incomes increases (which it must by the Furceri, Wodon-Yitzhaki theorem). The reader is hereby forewarned to be cautious about using log-incomes and their variances.

As shown in Table 2, we can find all possible combinations of Lorenz-worsening/improvement with convergent/divergent log-income changes. In particular, contrary to the share-change case, we can find examples that make compatible falling inequality as gauged by a Lorenz-improvement and divergent log-income changes.

The previous examples illustrate a more general point: that log-incomes can be divergent if a progressive transfer occurs sufficiently high up in the income distribution.

More precisely, we can show the following result for a single rank-preserving transfer that is sufficiently small:

Proposition 9 ***Log-income Panel Changes and Lorenz Dominance under a Single Rank-Preserving Transfer Sufficiently High Up in the Income Distribution***

Let gm denote the geometric mean of income at period 0, and $\exp(1) = 2.718$*. Consider two individuals i and j such that* $d_{i0} > d_{j0} > gm * \exp(1)$*. Let* $h > 0$ *be a sufficiently small rank-preserving transfer between i and j.*

a) If such a transfer h is equalizing, it produces a Lorenz-improvement $LC_1 \succ LC_0$*, rising inequality as gauged by the log-variance* $(V(\ln d_1) > V(\ln d_0))$*, and a divergent regression coefficient* $(\delta_{log} > 0)$*.*

b) If such a transfer h is disequalizing, it produces a Lorenz-worsening $LC_1 \prec LC_0$*, falling inequality as gauged by the log-variance* $(V(\ln d_1) < V(\ln d_0))$*, and a convergent regression coefficient* $(\delta_{log} < 0)$*.*

Proposition 9 suggests why it would be easy to misinterpret a log-change regression like (12). The log-change regression can indicate divergence as we define it, even when the income changes lead to a Lorenz-improvement. Rank-preserving equalizations which occur sufficiently high up in the income distribution can lead to divergence in log-dollars. This is an unappealing property of log-income regressions such as (12).

Exact Proportional Changes

As previously mentioned, one alternative to the log-income changes regression (12) is to regress the exact proportional change in incomes on initial income as in equation (5). In this case, we can establish results and conditions linking Lorenz-improvements/worsenings together with convergent/divergent exact proportional changes. In order to do this it is useful to define terms for *proportional* structural mobility and *proportional* exchange mobility:

$$\begin{aligned} PSM &= \frac{1}{n}\sum_i \frac{s_{ic} - s_{i0}}{s_{i0}} \\ PXM &= \frac{1}{n}\sum_i \frac{s_{i1} - s_{ic}}{s_{i0}}. \end{aligned} \tag{13}$$

Similar to the analysis of share changes, PSM is a term capturing the average proportional share changes due to changes in the shape of the income distribution if positions remain unchanged. In turn, PXM reflects proportional share changes associated with positional rearrangements, under a fixed marginal distribution. We can establish the following lemmas for these two terms, which mirror Lemmas 4 and 5.

Lemma 7 *Let PSM be given by equation (13), then:*

i) A Lorenz-improvement ($LC_1 \succ LC_0$) *implies* $PSM > 0$.

ii) A Lorenz-worsening ($LC_1 \prec LC_0$) *implies* $PSM < 0$.

Lemma 8

$$PXM = \frac{1}{n} \sum_i \frac{s_{i1} - s_{ic}}{s_{i0}} \geq 0.$$

With these lemmas established, we can show the following results linking inequality changes and exact proportional changes.

Proposition 10 ***Convergence in Exact Proportional Changes and Lorenz Dominance***

Let θ *be defined by the exact proportional change regression (5).*

i) A Lorenz-improvement ($LC_1 \succ LC_0$) *implies convergence in exact proportional changes* ($\theta < 0$).

ii) A Lorenz-worsening ($LC_1 \prec LC_0$) *is compatible with both convergent and divergent exact proportional changes.*

Also, as in the case of share and dollar changes, we can express part i) of the proposition as:

Corollary 5 *If the exact proportional changes are non-convergent* ($\theta \geq 0$)*, then either a weak Lorenz-worsening has taken place* ($LC_0 \succeq LC_1$)*, or the Lorenz curves of incomes in periods 0 and 1 cross.*

We can also establish the set of conditions giving all possible combinations in Table 8. The intuition is the same as before: if income changes are large enough, and in a suitable pattern, we can have positional changes, rising inequality, and convergent proportional changes all taking place at the same time.

A comparison of Table 8 with Table 6 shows that share-change and exact proportional change regressions share a similar structure with respect to Lorenz-dominance (something also apparent from Propositions 7 and 10). This stands in contrast to the comparison of dollar change and of log-dollar change regressions.

3.4 Special Cases

In this section, to gain additional understanding of how different patterns of Lorenz curve changes and divergent/convergent panel income changes can arise, we explore three special types of income changes. First, we consider the case where the income distribution vector does not change, but where individuals swap positions. Since there is no change in the anonymous distribution, the Lorenz curves and all inequality indices will remain unchanged. However, the positional swaps that occur will lead to a specific pattern of panel income changes.

The second case we consider is one in which all incomes change proportionately, i.e. all incomes are scaled-up or down by a constant factor κ. In this case, all relative inequality measures will remain the same, yet as we will show, some of our regression coefficients will be affected.

Finally, we consider the case when all individuals keep their same positions and yet there is a Lorenz-worsening. This case is of interest because, as we saw in the previous sections, it is not possible to rule out the existence of convergent panel income changes when inequality rises, due to the fact that there might be crossings among panel people as we go from one period to the other. It is then interesting to see whether in the absence of positional changes, rising relative inequality is or is not a sufficient condition for divergent panel changes.

The reason for considering such cases is that they are benchmarks against which to compare real-life examples. In particular the third case, of a Lorenz-worsening and no positional change, seems to be what many people have in mind when they think of rises in inequality.

In Table 9 we summarize our findings in each of these three special cases for each of the regressions used in the paper.

In the first column of that Table we show that in the case where the only income changes are due to positional changes then all of our regressions will exhibit convergence. In other words, in the absence of changes in the shape of the income distribution, the income changes due to rearrangements in positions will lead to convergence no matter how we measure it.

The second column indicates that a uniformly proportional increase in incomes is recorded by three of our panel regressions as neutral, i.e. neither convergent nor divergent. The one exception arises in the case of the dollar change regression. This exception for the dollar change regression occurs because a proportional increase(decrease) in dollars makes the dollar gains(losses) of the initially rich larger.

Finally, column three of Table 9 shows that in the absence of positional changes, a Lorenz-worsening leads to divergence for most of the regressions. In the case of the dollar change regression we need to assume the additional condition that the average income change in the economy is non-negative. Also in the case of the log-change regression, we have an ambiguous result because it depends where in the distribution the disequalizing income changes are taking place.[16]

In summary, the last column of Table 9 shows that in the absence of positional changes, a Lorenz-worsening leads usually but not always to divergent panel income changes. It is interesting to remark that this conclusion does not extend to other cases with rising relative inequality, but without a Lorenz-worsening. For instance, consider the transition:

$$[1, 2.99, 3] \rightarrow [2, 2.001, 5].$$

In this case there is a rise in inequality (the CV goes from 0.40 to 0.47), the Lorenz curves cross, there is no positional change, yet all of our regressions register convergence, rather than divergence.

3.5 Extensions to Cases Involving Single Lorenz Crossings

As previously noted, all the results in section 3.3 were derived by analyzing rising or falling inequality as judged by Lorenz-worsenings or improvements. Of course, it is possible for the Lorenz curves of two distributions to cross, which often happens in practice.[17] How far can we go allowing Lorenz curves

[16] It is easy to show that a similar set of results can be obtained in the case of a rise in Gini without positional changes (with or without a Lorenz-worsening), although in order to derive some of the results we might have to assume quadrant dependence as in section 3.2.2.

[17] See Atkinson (1973, 2008) for a classic discussion of the available evidence on Lorenz-crossings using real data in a cross-country setting.

to cross? When the curves cross, going from one distribution to the other involves some transfers that are equalizing and others that are disequalizing. Extending our earlier results to these cases is not straightforward. Our results for specific inequality indices apply, but we would like more general results.

Under certain circumstances we can do so. In particular, we require certain conditions on the nature of crossings of the Lorenz-curves, and the coefficients of variation of incomes. For example, if the Lorenz curve for final income crosses that of initial income once from above, then we have equalizing transfers toward persons at the low end and disequalizing transfers toward persons at the high end. For inequality (in a manner to be defined precisely below) to fall, it must be the case that the equalizing transfers "outweigh" the disequalizing ones. The condition that gives us this is a falling coefficient of variation.

We will follow the literature (Shorrocks and Foster, 1987) and restrict the class of inequality indices to those which are "transfer sensitive".

Definition 6 ***Transfer-sensitive Inequality Measures*** *(Shorrocks and Foster, 1987)*
*An inequality measure $I(d)$ is transfer sensitive **(TS)** iff $I(d_0) > I(d_1)$ whenever d_1 is obtained from d_0 by a series of transfers whereby at each stage i) a progressive transfer occurs at lower income levels, ii) a regressive transfer occurs at higher income levels, iii) ranks remain unchanged, and iv) the variance of incomes remains unchanged.*

Intuitively speaking, a Transfer-sensitive inequality measure is one that records a fall in inequality whenever there is a progressive transfer at the lower part of the income distribution in tandem with a regressive transfer at higher income levels, to the extent that the transfers are comparable in the sense of unchanged variance of incomes. An alternative way to state the transfer-sensitivity condition, is to require that the distribution of final incomes Third-Order stochastically dominate that of initial incomes.[18]

Transfer-sensitivity allows certain pairs of distributions to be ranked in the presence of Lorenz-crossings by giving greater weight to transfers that occur in the lower part of the income distribution. Shorrocks and Foster (1987) show that the Atkinson family and the Generalized Entropy class with parameter less than two satisfy the transfer-sensitive property, but the Gini coefficient does not.

[18]A formal statement together with a careful discussion of the concept is presented in Shorrocks and Foster (1987).

In this section, we will present a result for the following class of inequality measures.

Definition 7 I_{TS} ***Class of Inequality Measures***
Let $I_{TS}(d)$ *be the class of inequality measures satisfying transfer sensitivity (TS), the transfer principle (T), scale-independence (S), population-independence (P), and anonymity (A).*

We also formally define a single crossing from above.

Definition 8 ***Single Lorenz Crossing From Above***
Denote by $LC(d;p)$ *the Lorenz curve ordinate corresponding to the lowest 100p% of income recipients, for* $p \in [0,1]$. *The Lorenz curve for a distribution* d *is said to intersect that of* d' once from above *iff there exists* $p^* \in (0,1)$ *and intervals* $P \equiv [0,p^*]$ *and* $P' \equiv [p^*,1]$ *such that*

$$\begin{aligned} LC(d;p) &\geq LC(d';p) \quad \forall p \in P \quad \text{and } > \text{ for some } p \in P \\ LC(d;p) &\leq LC(d';p) \quad \forall p \in P' \quad \text{and } < \text{ for some } p \in P'. \end{aligned}$$

Intuitively, again, this entails equalizing transfers toward persons at the low end and disequalizing transfers toward persons at the high end.

With these definitions, we can now state results relating the I_{TS} class of inequality indices to the share change regressions.

Proposition 11 ***Convergence in Shares and Changes in Transfer Sensitive Inequality Indices under Single Lorenz-Crossing from Above***
If the Lorenz curve of d_1 *intersects that of* d_0 *once from above and* $CV(d_1) \leq CV(d_0)$, *then all measures in the* $I_{TS}(d)$ *class and the coefficients of the linear regressions of share changes* (δ_s) *are linked as follows:*

i) $I_{TS}(d_1) < I_{TS}(d_0)$

ii) $\delta_s < 0$.

Intuitively, Proposition 11 states that when the Lorenz curve of final incomes crosses that of initial incomes from above and the coefficient of variation falls or remains constant, then transfer sensitive inequality indices fall, and share changes are convergent. However, it bears mentioning that in this context share-convergence arises because of the changes in the CV, and not because of the particular type of Lorenz-crossing.

To empirically illustrate Proposition 11, consider the transition

$$d_0 = [1, 5, 10, 11] \to d_1 = [2, 4, 9, 12].$$

In this case the conditions of the Proposition are satisfied, namely there is: i) a single-crossing from above in the Lorenz curves, and ii) a falling CV (from 0.596 to 0.587). In this case it is readily verified that commonly used indices like the Atkinson family and Generalized Entropy with parameter < 2 will mark a reduction in inequality, and there is share convergence as well. More precisely, the Atkinson index with parameter 0.5 falls from 0.126 to 0.096 and the Generalized Entropy index with parameter 1 falls from 0.218 to 0.184. There is share convergence as well: in this situation, $\delta_s = -0.046$.

Finally, notice that Proposition 11 and Lemma 6 could be combined to derive conditions relating changes in Transfer-Sensitive inequality measures to convergence in dollar changes. This exercise is straightforward, and it is thus left for the reader.

This concludes our derivation of results. We turn now to a summary of the results and a concluding discussion.

4 Summary of Results and Concluding Observations

This paper has explored mathematically the relationship between changing relative income inequality in the cross section and panel income changes. All four combinations - rising inequality and convergent panel income changes, rising inequality and divergent panel income changes, falling inequality and convergent panel income changes, and falling inequality and divergent panel income changes - have been shown possible (Tables 1 and 2). The sources of results and conditions for each combination of rising or falling income inequality, convergent or divergent panel income changes, and positive or negative economic growth, have been derived in Section 3.

Four observations about the results are particularly trenchant.

First, all 32 cells in Table 2 are possible, provided inequality is suitably measured. But not all are possible measuring inequality change by comparing Lorenz curves.

Second, a large number of results are derived measuring inequality change by Lorenz-improvements and Lorenz-worsenings, which are particularly powerful criteria for making inequality comparisons. Thus, all who agree on the desirability of using Lorenz criteria for making inequality comparisons would feel confident that the various combinations involve "good" ways of measuring inequality.

Third, some of the results require Lorenz crossings or hold for a carefully chosen inequality index but not for all Lorenz-consistent indices. Consequently, these results are weaker than those based on Lorenz-dominance.

And fourth, some combinations are impossible, but there are only a few of them. One impossibility is the one previously proved by Furceri (2005) and Wodon and Yitzhaki (2006), namely that it is impossible to have divergent log-dollar changes and falling relative inequality as measured by the variance of logs. However, it is possible to have divergent log-dollar changes and a Lorenz-improvement, hence falling inequality as measured by *any Lorenz-consistent index*. The Furceri-Wodon-Yitzhaki impossibility result is due to the authors' use of an inequality index which is not Lorenz-consistent.

The other impossibilities are ones which we have proven here and were not previously in the literature. The various possibilities and impossibilities have been stated in Propositions 1 - 11, their Corollaries, and Tables 3 -8. The impossibilities arise when relative inequality falls according to a carefully chosen measure and when panel income changes are divergent in particular ways. Otherwise, we have proven that every other combination of rising or falling income inequality, divergent or convergent panel income changes, and economic growth or decline is possible, and we have displayed the conditions under which each arises.

To conclude let us return to where we started; namely with the reconciliation between i) convergent panel income changes and rising inequality, and between ii) divergent panel income changes and falling inequality.

The reason that convergent panel income changes can occur in spite of rising inequality, is that panel income changes can be large enough such that some initially low-earners become high earners in a widening distribution. In fact, it is precisely because panel studies abandon the property of single-period anonymity and replace it by two-period anonymity, that such a pattern can be identified. In particular, our paper shows how it is possible to have convergent dollar changes, together with rising relative inequality measures, in times of economic growth- a combination that is often observed in empirical data.

In addition, the coexistence of divergent panel income changes and falling inequality depends crucially on the way inequality and divergence are measured. In particular, we show that dollar changes can be divergent in conjunction with falling relative inequality measures if economy-wide income growth is positive and large enough. The reason for this is that a regression of dollar change on initial dollars is affected by proportional increases in income, while relative inequality measures are not.

The other instance where we can have divergence together with Lorenz-improvements occurs for regressions in log-dollars. More specifically, we showed that equalizing transfers that occur sufficiently high up in the income distribution will lead to a reduction in inequality by any measure that satisfies the transfer principle, yet the log-dollar regression will register divergence.

The results derived in this paper open up additional questions as to the empirical nature of individual income changes. For instance, when rising inequality is observed together with convergent panel income changes in empirical work, is this finding driven by a few individuals experiencing large changes, by many individuals experiencing moderate changes, or are both important? Exploring the precise way in which these large individual changes occur is an important question for future research.

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Tables

Table 1: Possibilities for Rising/Falling Inequality and Convergent/Divergent Panel Income Changes

	Falling Inequality	Rising Inequality
Convergent Panel Income Changes	✓	✓
Divergent Panel Income Changes	✓	✓

✓: This cell is possible both in times of economic growth and in times of economic decline.

Table 2: Matrix of Possibilities in Times of Economic Growth and Decline.

Final on Initial Regression: $y_1 = \alpha_y + \beta_y y_0 + u_y$
Changes Regression: $\Delta y_1 = \gamma_y + \delta_y y_0 + u_y$
Proportional Changes Regression: $\frac{d_1 - d_0}{d_0} = \phi + \theta d_0 + u_{pch}$

		Economic Growth Positive		Economic Growth Negative	
		Falling Relative Inequality	**Rising Relative Inequality**	**Falling Relative Inequality**	**Rising Relative Inequality**
Convergence/divergence	**Convergent**				
	Share changes ($\beta_s < 1 \iff \delta_s < 0$)	[5,20]→[10,20]^{LD}	[5,20]→[25,5]^{LD}	[5,25]→[5,20]^{LD}	[7,23]→[20,5]^{LD}
	Dollar changes ($\beta_d < 1 \iff \delta_d < 0$)	[5,20]→[10,20]^{LD}	[5,20]→[25,5]^{LD}	[5,25]→[5,20]^{LD}	[7,23]→[5,20]^{LD}
	Proportional changes *Log-dollar Approx.* ($\beta_{\mathbf{log}} < 1 \iff \delta_{\mathbf{log}} < 0$)	[5,20]→[10,20]^{LD}	[1,1,1,1,1,1,1,1,6.1,8.89]→ [1,1,1,1,1,1,1,1,6,9]^{LD}	[5,25]→[5,20]^{LD}	[1.1,407,418]→ [1,360,390]^{LD}
	Exact Prop. changes ($\theta < 0$)	[5,20]→[10,20]^{LD}	[5,20]→[25,5]^{LD}	[5,25]→[5,20]^{LD}	[7,23]→[20,5]^{LD}
	Divergent				
	Share changes ($\beta_s > 1 \iff \delta_s > 0$)	[1,5,10]→ [2,4,25]*	[5,20]→[5,25]^{LD}	[60,320,1000]→ [54,150,876]*	[10,20]→[5,20]^{LD}
	Dollar changes ($\beta_d > 1 \iff \delta_d > 0$)	[5,20]→[7,23]^{LD}	[5,20]→[5,25]^{LD}	[20,90,180]→ [20,61,180]*	[10,20]→[5,20]^{LD}
	Proportional changes *Log-dollar Approx.* ($\beta_{\mathbf{log}} > 1 \iff \delta_{\mathbf{log}} > 0$)	[1,360,390]→ [1.1,407,418]^{LD}	[5,20]→[5,25]^{LD}	[1,1,1,1,1,1,1,1,6,9]→ [1,1,1,1,1,1,1,1,6.1,8.89]^{LD}	[10,20]→[5,20]^{LD}
	Exact Prop. changes ($\theta > 0$)	[1,5,10]→ [2,4,25]*	[5,20]→[5,25]^{LD}	[60,320,1000]→ [54,150,876]*	[10,20]→[5,20]^{LD}

Notes: LD: Lorenz-Dominance
*: Possible when measure changing inequality by income share of the poorest tercile, because Lorenz curves cross. Lorenz-dominance is not possible in this cell.

Table 3: Conditions for Convergent/Divergent Income Changes and Falling/Rising Variances

	Falling Variance	Rising Variance
Convergent Panel Income Changes	$V(\Delta y) < 2\lvert cov(\Delta y, y_0)\rvert$ and $cov(\Delta y, y_0) < 0$	$0 < \Delta V(y) < V(\Delta y)$
Divergent Panel Income Changes	Impossible	$\Delta V(y) > V(\Delta y)$

For $y = f(d)$, and $f(\cdot)$ any monotonically increasing function of income in dollars.

Table 4: Conditions for Convergent/Divergent Dollar Changes and Falling/Rising Coefficient of Variation in Times of Growth and Decline

A: Economic Growth ($g > 0$)

	Falling CV	Rising CV
Convergent Dollar Changes	$\rho_d(1+g) < \frac{CV_0}{CV_1}$ and $CV_1 < CV_0$	$\rho_d(1+g) < \frac{CV_0}{CV_1} < 1$
Divergent Dollar Changes	$\rho_d(1+g) > \frac{CV_0}{CV_1} > 1$	$\rho_d(1+g) > \frac{CV_0}{CV_1}$ and $CV_0 < CV_1$

B: Economic Decline ($g < 0$)

	Falling CV	Rising CV
Convergent Dollar Changes	$\rho_d(1+g) < 1 < \frac{CV_0}{CV_1}$	$\rho_d(1+g) < \frac{CV_0}{CV_1} < 1$
Divergent Dollar Changes	Impossible	$1 > \rho_d(1+g) > \frac{CV_0}{CV_1}$

Where ρ_d is the correlation coefficient between initial and final dollars, CV_t is the coefficient of variation of incomes in dollars in period t, and g is the average rate of growth in the economy.

Table 5: Conditions for Convergent/Divergent Share Changes and Gini Changes

	Falling Gini	Rising Gini
Convergent Share Changes	$E(s_1\Delta r) < \|cov(\Delta s, r_0)\|$ and $cov(\Delta s, r_0) < 0$	$E(s_1\Delta r) > \|cov(\Delta s, r_0)\|$ and $cov(\Delta s, r_0) < 0$
Divergent Share Changes	Impossible	$\Delta G > 2E(s_1\Delta r)$

G: Gini index, r_0: rank when distribution is sorted according to initial-period income.
s_t: income shares in period t.
Share convergence/divergence is gauged through a regression of share changes on initial ranks (4).

Table 6: Conditions for Convergent/Divergent Share Changes under Lorenz Dominance

	Lorenz-Improvement	Lorenz-Worsening
Convergent Share Changes	$SM < 0$	$\|XM\| > SM > 0$
Divergent Share Changes	Impossible	$SM > \|XM\| \geq 0$

SM: Structural Mobility. XM: Exchange Mobility.
For definitions see equation (9).

Table 7: Conditions for Convergent/Divergent Dollar Changes under Lorenz Dominance in Times of Growth and Decline

A: Economic Growth ($g > 0$)

	Lorenz-Improvement	Lorenz-Worsening
Convergent Dollar Changes	$SM < 0$ and $\beta_s(1+g) < 1$	$\|XM\| > SM > 0$ and $\beta_s(1+g) < 1$
Divergent Dollar Changes	$SM < 0$ and $1 < \beta_s(1+g)$	$SM > 0$ and $1 < \beta_s(1+g)$

B: Economic Decline ($g < 0$)

	Lorenz-Improvement	Lorenz-Worsening
Convergent Dollar Changes	$SM < 0$	$SM > 0$ and $\beta_s(1+g) < 1$
Divergent Dollar Changes	Impossible	$SM > \|XM\| \geq 0$ and $1 < \beta_s(1+g)$

SM: Structural Mobility. XM: Exchange Mobility.
For definitions see equation (9).

Table 8: Conditions for Convergent/Divergent Exact Proportional Changes under Lorenz Dominance

	Lorenz-Improvement	Lorenz-Worsening
Convergent Exact Proportional Changes	$PSM > 0$	$PXM > \|PSM\| > 0$ and $PSM < 0$
Divergent Exact Proportional Changes	Impossible	$\|PSM\| > PXM \geq 0$ and $PSM < 0$

PSM: Proportional Structural Mobility.
PXM: Proportional Exchange Mobility.
For definitions see equation (13).

Table 9: Special Cases

	Lorenz Curves remain unchanged		Lorenz Worsening & No Positional Change
	Positional Change Only	Uniform Proportional Change	
Share Changes	Convergent	Neutral	Divergent
Dollar Changes	Convergent	Convergent if $g < 0$ Divergent if $g > 0$	Divergent if $g \geq 0$
Proportional Changes			
Log Dollar Approx.	Convergent	Neutral	Ambiguous
Exact Prop. changes	Convergent	Neutral	Divergent

Proofs of Results

In what follows, we will assume a finite population.

Proposition 1 ***Convergence and Changes in Variance for the Class of Monotonic Transformations of Income in Dollars.***
Let $f(d)$ be any monotonically increasing function of income in dollars and denote the value of this function by y. Consider the change-on-initial regression (3.2). Then:

i) If $\Delta V(y) < 0$, then $\delta_y < 0$, i.e., the changes in generic income are convergent.

ii) If $\Delta V(y) > 0$, the changes in y can be either convergent or divergent.

Proof of Proposition 1. First express the regression (3.2) in its final-on-initial form (3.1)

$$y_1 = \alpha_y + (\delta_y + 1)y_0 + u_y. \tag{3.1}$$

Take the variance of both sides

$$V(y_1) = (\delta_y + 1)^2 V(y_0) + V(u_y),$$

since $cov(y_0, u_y) = 0$ because (3.1) is a linear projection.

Note we can rewrite the change in variances of y as

$$\Delta V(y) = V(y_1) - V(y_0) = \delta_y(\delta_y + 2)V(y_0) + V(u_y). \tag{A.1}$$

If the left-hand side of (A.1) is non-positive, and we exclude the extreme cases where $V(y_0) = 0$, then it must be the case that $-2 < \delta_y \leq 0$. This proves part i).

Part ii) follows immediately from equation (A.1). To illustrate that indeed both convergence and divergence can arise under $\Delta V(y) > 0$ consider the following example.

Assume there are two individuals with indices $i \in \{1, 2\}$. Furthermore, assume

$$y_{10} = y_{21} < y_{20} < y_{11}.$$

In other words, initially individual 1 was poorer than individual 2. However, in period 2, the initially poor individual becomes the richest with a resulting income (y_{11}) greater than the first-period's income of the initially rich individual (y_{20}). For simplicity, we assume that this initially rich individual ends up in period two with a low income of y_{10}.

By definition, in period t, $V(f(d_t))$ equals

$$V(f(d_t)) = \frac{1}{2}(y_{1t}^2 + y_{2t}^2) - \overline{y}_t^2$$

hence,

$$\Delta V(f(d_t)) = \frac{1}{2}(y_{11}^2 + y_{21}^2 - y_{10}^2 - y_{20}^2) - (\overline{y}_1^2 - \overline{y}_0^2).$$

Since in our example we assumed $y_{10} = y_{21}$, after some algebra the previous expression simplifies to

$$\Delta V(f(d_t)) = \frac{y_{11} - y_{20}}{4}(y_{11} + y_{20} - 2y_{10}) > 0,$$

where, again, the sign of the change in the variance is determined by our initial assumptions on the values of y_{it}.

We have then established that our exaple satisfies the premise of part iii), namely, the variance of y rises. Now we need to show that for any regression

$$\Delta y = \gamma_y + \delta_y y_0 + u_y$$

our example yields $\delta_y < 0$.

The sign of δ_y is given by

$$\begin{aligned} cov(\Delta y, y_0) = \frac{1}{2}\left((y_{11} - y_{10})f_{10} + (y_{21} - y_{20})y_{20}\right) \\ - \frac{(y_{11} - y_{10}) + (y_{21} - y_{20})}{2}\frac{y_{10} + y_{20}}{2}. \end{aligned}$$

Again, using our simplifying assumption $y_{10} = y_{21}$, after some algebra the previous expression simplifies to

$$cov(\Delta y, y_0) = \frac{y_{11} + y_{20} - 2y_{10}}{4}(y_{10} - y_{20}) < 0,$$

where, again, the sign of the covariance is determined by our initial assumptions on the values of y_{it}.

Hence, we have established that in our example there is a compatiblity between a rising variance of y and convergent changes in y.

To complete the illustration we need to show that it is possible to generate examples with a rising variance of y and divergent changes of y. To show this example modify our previous assumption to be

$$y_{10} = y_{11} < y_{20} < y_{21}.$$

In other words, the individual 1 remain the poorest across both periods, and experienced no income change. However, in period 2, the initially rich individual became even richer. It is easy to show that there is a rise in the variance of y. To establish divergence notice that under the stated assumption we have

$$\begin{aligned} cov(\Delta y, y_0) &= \frac{(y_{21} - y_{20})y_{20}}{2} - \frac{(y_{21} - y_{20})}{2}\frac{y_{10} + y_{20}}{2} \\ &= \frac{(y_{21} - y_{20})y_{20}}{2} - \frac{(y_{20} - y_{10})}{2} > 0. \end{aligned}$$

In summary, when the variance of y rises it is possible to generate examples that imply divergent or convergent changes in y. □

Lemma 1 ***Variance of Shares and Coefficient of Variation***
Let $CV(d)$ denote the coefficient of variation of income, then

$$CV^2(d) = V(s).$$

Proof of Lemma 1.

$$V(s) = V\left(\frac{d}{\mu}\right) = \frac{1}{\mu^2}V(d) = CV^2(d).$$

□

Proposition 2 ***Convergence in Dollars, Changes in the Coefficient of Variation, and Economic Growth.***

Let β_d be defined by the final-on-initial regression (3.1) when income is measured in dollars, and denote the correlation coefficient from this regression *by ρ_d. Let $CV(d_t)$ denote the coefficient of variation of income at period t, and let g denote the economy-wide growth rate in incomes between year 0 and year 1. Then there is divergence/convergence in dollars as follows:*

$$\beta_d \gtreqless 1 \qquad \text{(i.e. } \delta_d \gtreqless 0) \iff \rho_d \frac{CV(d_1)}{CV(d_0)}(1+g) \gtreqless 1. \tag{8}$$

Proof of Proposition 2. By definition

$$\rho_d = \frac{cov(d_1, d_0)}{\sqrt{V(d_1)}\sqrt{V(d_0)}}$$

and

$$\beta_d = \rho_d \frac{\sqrt{V(d_1)}}{\sqrt{V(d_0)}}.$$

However,

$$\begin{aligned} \frac{\sqrt{V(d_1)}}{\sqrt{V(d_0)}} &= \frac{\sqrt{V(d_1)}/\mu_1}{\sqrt{V(d_0)}/\mu_0} \frac{\mu_1}{\mu_0} \\ &= \frac{CV(d_1)}{CV(d_0)} \frac{\mu_1}{\mu_0}. \end{aligned}$$

Moreover,

$$\mu_1 = (1+g)\mu_0$$

where g is the economy-wide income growth rate. Combining these equations together we obtain equation (8). □

Lemma 2 ***Share Changes and Exact Proportional Changes***

Let θ be the slope of the exact proportional change regression (5) *then:*

$$sign(\theta) = -sign\left(E\left[\frac{s_1 - s_0}{s_0}\right]\right).$$

Proof of Lemma 2. First, rewrite the proportional change regression (5) as

$$\frac{d_1}{d_0} = (\phi + 1) + \theta d_0 + u_{pch}.$$

Then the sign of θ will depend on the sign of the covariance

$$\begin{aligned} cov\left(\frac{d_1}{d_0}, d_0\right) &= E\left(\frac{d_1}{d_0} d_0\right) - E\left(\frac{d_1}{d_0}\right)\mu_0 \\ &= \mu_1 - E\left(\frac{d_1}{d_0}\right)\mu_0. \end{aligned}$$

Hence, there will be divergence (i.e. $\theta > 0$) whenever $\mu_1 > E(\frac{d_1}{d_0})\mu_0$, convergence (i.e. $\theta < 0$) whenever $\mu_1 < E(\frac{d_1}{d_0})\mu_0$, otherwise the profiles will be parallel.

This condition for convergence can be re-expressed as

$$\begin{aligned} E\left(\frac{d_1}{d_0}\right)\mu_0 - \mu_1 &> 0 \\ E\left(\frac{d_1}{d_0}\right)\frac{\mu_0}{\mu_1} - 1 &> 0 \\ E\left(\frac{s_1}{s_0}\right) - 1 &> 0. \end{aligned}$$

So we can express these conditions as:

Convergence ($\theta < 0$) $\iff 0 < E[\frac{s_1 - s_0}{s_0}]$

Divergence ($\theta > 0$) $\iff 0 > E[\frac{s_1 - s_0}{s_0}]$

Parallel Profiles ($\theta = 0$) $\iff 0 = E[\frac{s_1 - s_0}{s_0}]$ □

Lemma 3 ***Properties of Quadrant-Dependent Variables***

Let (X, Y) be a bivariate random variable. Let $f(X)$ be a strictly monotonically increasing function of X. Similarly, let $g(X)$ be a strictly monotonically decreasing function of X. If (X, Y) is quadrant-dependent, then

i) $sign(cov(X, Y)) = sign(cov(f(X), Y)$

ii) $sign(cov(X, Y)) = -sign(cov(g(X), Y)$,

provided the covariances exist.

Proof. **Proof of Lemma 3**

Let X and Y have marginal cdf's $F_x(x)$ and $F_y(y)$, respectively, and joint bivariate cdf $H(x, y)$. Denote by $H_f(f_x, y)$ the joint bivariate cdf of $(f(X), Y)$, i.e.,

$$H_f(f_x, y) = P(f(X) \leq f_x, Y \leq y).$$

Also, let $F_f(f_x)$ be the marginal cdf of $f(X)$. Define in a similar manner $H_g(g_x, y)$ and $F_g(g_x)$, for $(g(X), Y)$.

Finally, let $f^{-1}(\cdot)$ and $g^{-1}(\cdot)$ be the inverse of $f(X)$ and $g(X)$, respectively.

Let us first establish the following results:

a) If (X, Y) is positively (negatively) quadrant-dependent, then $(f(X), Y)$ is positively (negatively) quadrant-dependent.

b) If (X, Y) is positively (negatively) quadrant-dependent, then $(g(X), Y)$ is negatively (positively) quadrant-dependent.

To establish a) notice that positive quadrant-dependence of (X, Y) implies

$$H(f^{-1}(f_x), y) \geq F_x(f^{-1}(f_x))F_y(y) \quad \forall f^{-1}(f_x), y,$$

yet,

$$F_x(f^{-1}(f_x)) = P(X \leq f^{-1}(f_x)) = P(f(X) \leq f_x) = F_f(f_x),$$
$$H(f^{-1}(f_x), y) = P(X \leq f^{-1}(f_x), Y \leq y) = P(f(X) \leq f_x, Y \leq y) = H_f(f_x, y).$$

Hence, positive quadrant-dependence of (X, Y) implies

$$H_f(f_x, y) \geq F_f(f_x)F_y(y) \quad \forall f_x, y,$$

i.e. positive quadrant-dependence of $(f(X), Y)$. A similar argument establishes the negatively quadrant-dependence case.

To establish b) notice that for a strictly monotonically decreasing transformation $g(X)$,

$$P(X \geq g^{-1}(g_x)) = P(g(X) \leq g_x) = F_g(g_x)$$
$$P(X \geq g^{-1}(g_x), Y \leq y) = P(g(X) \leq g_x, Y \leq y) = H_g(g_x, y).$$

By Lemma 1(v)(2.1") in Lehmann (1966), we know that

$$P(X \geq g^{-1}(g_x), Y \leq y) \leq P(X \geq g^{-1}(g_x))P(Y \leq y) \quad \forall g^{-1}(g_x), y,$$

is equivalent to positive quadrant-dependence of (X, Y). Hence, positive quadrant-dependence of (X, Y) implies

$$H_g(g_x, y) \leq F_g(g_x) F_y(y),$$

i.e., negative quadrant-dependence of $(g(X), Y)$.

A similar argument establishes that negative quadrant-dependence of (X, Y) implies positive quadrant-dependence of $(g(X), Y)$.

Furthermore, recall a useful result by Hoeffding (1940, 1994), which establishes that for two random variables (X, Y),

$$cov(X, Y) = \int_{\mathbb{R}^2} (H(x, y) - F_x(x) F_y(y)) dx dy,$$

provided $E(|XY|)$, $E(|X|)$, and $E(|Y|)$ are all finite.

This result implies that positive quadrant-dependence between any two variables (X, Y) implies $cov(X, Y) \geq 0$. Similarly, negative quadrant-dependence between (X, Y) implies $cov(X, Y) \leq 0$, provided the covariances exist.

Applying these results to the case a) above means that if (X, Y) is positively quadrant-dependent then both $cov(X, Y) \geq 0$ and $cov(f(X), Y) \geq 0$, and if (X, Y) is negatively quadrant-dependent then both $cov(X, Y) \leq 0$ and $cov(f(X), Y) \leq 0$, provided the covariances exist.

Similarly, for the case b), if (X, Y) is positively quadrant-dependent then both $cov(X, Y) \geq 0$ and $cov(g(X), Y) \leq 0$, and if (X, Y) is negatively quadrant-dependent then both $cov(X, Y) \leq 0$ and $cov(g(X), Y) \geq 0$, provided the covariances exist.

It only remains to establish that

$$cov(X, Y) = 0 \iff cov(f(X), Y) = 0 \iff cov(g(X), Y) = 0.$$

However, Lemma 3 in Lehmann (1966) shows that under quadrant-dependence $cov(X, Y) = 0$ if and only if X and Y are independent. Yet, if X and Y are independent, so are $f(X)$ and Y, and $g(X)$ and Y.

This establishes the last equivalences and proves the Lemma. □

Proposition 3 ***Convergence in Exact Proportional Changes and Changes in the Coefficient of Variation***

Let θ be defined by the exact proportional change regression (5). Assume that the bivariate random variable, $(s_0, \Delta s)$, is quadrant-dependent. Then:

i) If $\Delta CV(d) < 0$, then $\theta < 0$, i.e., the exact proportional changes are convergent.

ii) If $\Delta CV(d) > 0$, the exact proportional changes can be either convergent or divergent.

Proof of Proposition 3. From Proposition 1 for the case of shares and Lemma 1, it follows that

$$\Delta CV(d) < 0 \Rightarrow cov(\Delta s, s_0) < 0.$$

Since $1/s_0$ is strictly decreasing in s_0, by Lemma 3.ii), under quadrant-dependence of $(s_0, \Delta s)$, $cov(\Delta s, s_0)$ and $cov(\Delta s, 1/s_0)$ have opposite signs, namely

$$cov(\Delta s, 1/s_0) > 0,$$

provided the covariances exist.

However, $0 < cov(\Delta s, 1/s_0)$ implies

$$0 < E\left[\frac{s_1 - s_0}{s_0}\right],$$

since $E(\Delta s) = 0$.

Finally, from Lemma 2 it follows that:

$$0 < E\left[\frac{s_1 - s_0}{s_0}\right] \iff \theta < 0.$$

In other words, under the stated assumptions, δ_s and θ share the same sign. This proves part i).

To establish part ii) notice that a rising coefficient of variation, $\Delta CV(d) > 0$, implies a rising variance of shares, $\Delta V(s) > 0$, by Lemma 1. However, recall from our proof of Proposition 1 that

$$\Delta V(s) = \delta_s(\delta_s + 2)V(s_0) + V(u_s).$$

For $\delta_s > 0$, $\Delta V(s) > 0$ automatically. Nevertheless, for certain negative values of δ_s it will still be the case $\Delta V(s) > 0$. Since under the stated assumptions δ_s and θ share the same sign, this means that a rising coefficient of variation is compatible with either convergence or divergence in exact proportional changes.

As a form of illustration, the transition

$$[1, 3] \to [5, 1]$$

is an example of a rising coefficient of variation, and convergent exact proportional changes, i.e. $\theta < 0$. While the transition

$$[1, 3] \to [1, 5]$$

is an example of a rising coefficient of variation, and divergent exact proportional changes, i.e. $\theta > 0$.

□

Proposition 4 ***Share-on-Ranks Convergence and Changes in the Gini***

Let G_t be the Gini index in period t, and let λ be given by equation (4). Then:

i) If $\Delta G < 0$, then $\lambda < 0$, i.e., the share-on-rank changes are convergent.

ii) A rising Gini may be consistent with either convergent or divergent share-on-rank changes as measured by λ.

Proof of Proposition 4. We can express the Gini index at time t as

$$G_t = -\frac{n+1}{n} + \frac{2}{n}\sum_{i=1}^{n} r_{it}s_{it}.$$

Hence, the change in Ginis can be expressed as

$$\begin{aligned} G_1 - G_0 &= \frac{2}{n}\sum_{i=1}^{n}(r_{i1}s_{i1} - r_{i0}s_{i0}) \\ &= \frac{2}{n}\sum_{i=1}^{n}(r_{i1}s_{i1} - r_{i0}s_{i1} + r_{i0}s_{i1} - r_{i0}s_{i0}) \\ &= \frac{2}{n}\sum_{i=1}^{n}[(r_{i1} - r_{i0})s_{i1} + r_{i0}(s_{i1} - s_{i0})]. \end{aligned}$$

In other words, we arrive at the following decomposition

$$G_1 - G_0 = \frac{2}{n}\sum_{i=1}^{n}(s_{i1}\Delta r_i + r_{i0}\Delta s_i). \tag{A.2}$$

First we need to establish the following Lemma.

Lemma

$$\sum_{i=1}^{n} s_{i1}\Delta r_i \geq 0.$$

Proof. Order the individuals in ascending order of initial income shares, and create an $n \times n$ matrix A, whose rows and columns identify the individuals in that order- same order for rows and columns.

Let the entries of the A matrix be filled as follows: if individual p overtakes individual q (i.e., p's rank is less than q's initially, but greater in final shares), then $A(p,q) = 1$ and $A(q,p) = -1$. That is, $A(p,q) = 1$ if p's share of final income exceeds that of q when p's share of initial income was less than q's. If two individuals do not overtake one another then $A(q,p) = 0$. By constructing the matrix A in this manner we ensure that for all $i, j \in \{1, \dots, n\}$ it is the case that $A(i,j) = -A(j,i)$.

From this construction, it follows that we can express the change in ranks for any given individual as the column sum for a given row of matrix A, i.e.,

$$\Delta r_i = \sum_{q=1}^{n} A(i,q).$$

Hence we can write,

$$\sum_{i=1}^{n} s_{i1}\Delta r_i = \sum_{i=1}^{n} s_{i1}\left(\sum_{q=1}^{n} A(i,q)\right).$$

This sum aggregates terms of the form $s_{i1}A(i,j)$ for three types of pairs (i,j):

a) $Z = \{i, j \in (1, \dots, n) | A(i,j) = 0\}$,

b) $Pos = \{i, j \in (1, \dots, n) | A(i,j) = 1\}$,

c) $Neg = \{i, j \in (1, \dots, n) | A(i,j) = -1\}$.

For each element in the set Pos there is a corresponding element in the set Neg (since $A(i,j) = -A(j,i)$), so we can now sum over the pairs

$$\begin{aligned}\sum_{i=1}^{n} s_{i1}\Delta r_i &= \sum_{(p,q)\notin Z} [s_{p1}A(p,q) + s_{q1}A(q,p)] \\ &= \sum_{(p,q)\notin Z} (s_{p1} - s_{q1})\end{aligned}$$

yet we know that $s_{p1} > s_{q1}$ since p overtook q going up, hence we have a sum of positives (or zeroes if there was no positional change). $\square$

The intuition behind this result is that for any upward rank change there will be one or more downward rank changes such that the overall sum of the upward and downward rank changes is zero. The upward rank change is multiplied by a larger *final* income share than are the downward rank changes. This is true for all upward rank changes, individually and together.

As a consequence of this last Lemma we can establish that if $\Delta G < 0$, it must be that the second term

$$\frac{2}{n}\sum_{i=1}^{n} r_{i0}\Delta s_i$$

is negative.

This term however, is a (rescaled) covariance between share changes and initial ranks. In particular,

$$\begin{aligned}cov(\Delta s, r_0) &= E(r_0\Delta s) - E(r_0)E(\Delta s) \\ &= E(r_0\Delta s),\end{aligned}$$

as $E(\Delta s) = 0$, by construction. If this term is negative (as it is when $\Delta G < 0$) then $\lambda < 0$, since by definition

$$\lambda = \frac{cov(\Delta s, r_0)}{V(r_0)}.$$

This proves part i).

Finally, to prove ii) notice that by virtue of the decomposition (A.2), in the case with positional changes, the sign of ΔG is determined by the sum of a term that is strictly positive (i.e. $E(s_1\Delta r)$) and another one ($cov(\Delta s, r_0)$) that can be either positive or negative.

If the Gini is rising, then the RHS of (A.2) is positive and there are no restrictions on the sign of $cov(\Delta s, r_0)$, and hence of λ.

As a form of illustration, the transition

$$[1,3] \rightarrow [5,1]$$

is an example of a rising Gini index, and convergent share-on-rank changes, i.e. $\lambda < 0$. While the transition

$$[1,3] \rightarrow [1,5]$$

is an example of a rising Gini index, and divergent share-on-rank changes, i.e. $\lambda > 0$. □

Proposition 5 ***Convergence in Dollars, Changes in the Gini, and Economic Growth***

Let δ_d be defined by the regression of change in dollars on initial dollars (7), and let g denote the economy-wide growth rate in incomes between year 0 and year 1. Assume that the bivariate random variable, $(r_0, \Delta s)$ is quadrant-dependent. Then:

i) If $\Delta G < 0$ and $g \leq 0$, then $\delta_d < 0$, i.e., the dollar changes are convergent.

ii) A rising Gini may be consistent with either convergent or divergent dollar changes as measured by δ_d.

Proof of Proposition 5. Notice first that using the identity

$$\mu_1 \Delta s = \Delta d - g d_0,$$

we can re-express the covariance $cov(\Delta s, d_0)$ as:

$$\begin{aligned} cov(\Delta s, d_0) &= E(d_0 \Delta s) - E(\Delta s)E(d_0) \\ &= E(d_0 \Delta s) \\ &= \mu_1^{-1} E[d_0(\Delta d - g d_0)] \\ &= \mu_1^{-1}[E(d_0 \Delta d) - gE(d_0^2)]. \end{aligned}$$

Furthermore, we can write $cov(\Delta d, d_0)$ as

$$\begin{aligned} cov(\Delta d, d_0) &= E(d_0 \Delta d) - E(\Delta d)E(d_0) \\ &= E(d_0 \Delta d) - g\mu_0^2 \\ &= E(d_0 \Delta d) - gE(d_0^2) + gE(d_0^2) - g\mu_0^2 \\ &= [E(d_0 \Delta d) - gE(d_0^2)] + gV(d_0) \\ &= cov(\Delta s, d_0)\mu_1 + gV(d_0). \end{aligned} \tag{A.3}$$

The proof of the various parts is established by using this last equation and noting that the transformation from initial ranks r_0 to initial dollars d_0 is strictly increasing, so by Lemma 3.i), under quadrant-dependence of $(r_0, \Delta s)$, $cov(\Delta s, r_0)$ and $cov(\Delta s, d_0)$ share the same sign.

Proof of part i):

From Proposition 4 we know that a falling Gini implies $\lambda < 0$, therefore $cov(\Delta s, r_0) < 0$. Under the quadrant-dependence assumption this in turn implies $cov(\Delta s, d_0) < 0$, provided the covariances exist.

If in addition, $g \leq 0$, then we ensure $cov(\Delta d, d_0) < 0$ and so there will be convergence in dollars, i.e. $\delta_d < 0$.

Part ii) can be established by example. The transition

$$[1, 3] \to [5, 1]$$

is an example of a rising Gini index, and convergent dollar changes, i.e. $\delta_d < 0$. In contrast, the transition

$$[1, 3] \to [1, 5]$$

is an example of a rising Gini index, and divergent dollar changes, i.e. $\delta_d > 0$.

□

Proposition 6 ***Convergence in Exact Proportional Changes and Changes in the Gini***

Let θ be defined by the exact proportional change regression (5). Assume that the bivariate random variable $(r_0, \Delta s)$ is quadrant-dependent. Then:

i) If $\Delta G < 0$, then $\theta < 0$, i.e., the exact proportional changes are convergent.

ii) A rising Gini may be consistent with either convergent or divergent exact proportional changes as measured by θ.

Proof of Proposition 6. To establish this proposition note that by Lemma 3.ii), under quadrant-dependence of $(r_0, \Delta s)$, $cov(\Delta s, r_0)$ and $cov(\Delta s, 1/s_0)$ have opposite signs, since $1/s_0$ is strictly decreasing in r_0.

However,

$$cov(\Delta s, 1/s_0) = E\left[\frac{s_1 - s_0}{s_0}\right]$$

since $E(\Delta s) = 0$.

Finally, from Lemma 2 it follows that:

$$sign\left(E\left[\frac{s_1 - s_0}{s_0}\right]\right) = -sign(\theta).$$

Since the sign of λ is determined by $cov(\Delta s, r_0)$, under the stated assumptions λ and θ share the same sign.

Proof of part i) Whenever $\Delta G < 0$ then $cov(\Delta s, r_0) < 0$ by Proposition 4.i). By the above argument, $\theta < 0$, provided the covariances exist.

Part ii) can be proven by example. In particular, consider the transition

$$[1, 3] \to [5, 1]$$

is an example of a rising Gini index, and convergent exact proportional changes, i.e. $\theta < 0$. In contrast, the transition

$$[1, 3] \to [1, 5]$$

is an example of a rising Gini index, and divergent exact proportional changes, i.e. $\theta > 0$.

$\square$

Lemma 4 *Let SM be given by equation (9),*

$$SM = \frac{\sum_i (s_{ic} - s_{i0}) s_{i0}}{n},$$

then:

i) A Lorenz-improvement $(LC_1 \succ LC_0)$ *implies* $SM < 0$.

ii) A Lorenz-worsening $(LC_1 \prec LC_0)$ *implies* $SM > 0$.

Proof of Lemma 4. Proof of part ii)

Let s_0 be the initial vector of shares and let s_c be defined as in (1).

Theorem 2.1 in Fields and Fei (1978) implies that if the distribution of s_0 Lorenz-dominates that of s_c, i.e. if $LC_0 \succ LC_c$, then it is possible to go from s_0 to s_c by means of a sequence of rank-preserving disequalizing transfers.

One convenient way of representing such transfers is by indexing them as $h(i,j)$ where the first argument, i, indicates which individual is making a transfer and the second one, j, which one is receiving it.

Since the transfers are disequalizing, and no one makes a transfer to himself, they satisfy the conditions:

$$\begin{aligned} &h(i,j) = 0 \quad \text{for } d_{i0} \geq d_{j0} \\ &h(i,j) \geq 0 \quad \text{for } d_{i0} < d_{j0} \quad \text{with strict inequality for some pair } \{\text{i,j}\}. \end{aligned}$$

The total transfers made by individual i will be the sum over the second index j, namely

$$h(i,\cdot) = \sum_{j=1}^{n} h(i,j).$$

Similarly, the total transfers received by this same individual will be the sum over the first index, namely

$$h(\cdot,i) = \sum_{j=1}^{n} h(j,i).$$

Hence, the change in this person's income share can be expressed as the difference between the two previous quantities, i.e.

$$s_{ic} - s_{i0} = h(\cdot,i) - h(i,\cdot) = \sum_{j=1}^{n} h(j,i) - \sum_{j=1}^{n} h(i,j).$$

By construction, the sum of the share changes over all individuals is zero, hence each person's share loss is somebody else's share gain, and also each share gain is somebody else's loss. In other words, any given transfer $h(i,j)$ appears with a positive sign in the share change of individual j, and with a negative sign in the share change of individual i. Furthermore, at any given stage of the sequence of transfers, the sender i is always poorer than the receiver j, since the transfer is disequalizing. Hence, for each transfer $h(i,j)$ we have

$$h(i,j)(\tilde{s}_j - \tilde{s}_i) \geq 0$$

where $\tilde{s}_i$ and $\tilde{s}_j$ are the shares of individuals i and j, respectively, at the given stage of the sequence of transfers where $h(i,j)$ takes place.

Since each of these transfers are rank-preserving, it follows that at any given stage of the sequence of transfers, $\tilde{s}_j - \tilde{s}_i \geq 0$ implies

$$h(i,j)(s_{j0} - s_{i0}) \geq 0.$$

Notice however, that SM can be rewritten as

$$\begin{aligned} SM &= n^{-1} \sum_i (s_{ic} - s_{i0}) s_{i0} \\ &= n^{-1} \sum_i \left(\sum_{j=1}^{n} h(j,i) - \sum_{j=1}^{n} h(i,j) \right) s_{i0}. \end{aligned}$$

That is, SM will be the average of terms $h(i,j)(s_{j0} - s_{i0})$ for all the transfers $h(i,j)$. Since these terms are non-negative, and some will be strictly positive, then SM will be positive.

In other words, we have shown that $LC_0 \succ LC_c$ implies $SM > 0$. However, by construction, the Lorenz curve of the vector s_c is the same as that of the final income vector s_1 (i.e. $LC_c = LC_1$), so we have that a Lorenz-worsening $LC_0 \succ LC_1$ implies $SM > 0$.

The proof of part i) follows by reproducing the previous steps, now with rank-preserving equalizing transfers. $\square$

Lemma 5

$$XM = \frac{\sum_i (s_{i1} - s_{ic}) s_{i0}}{n} \leq 0.$$

Proof of Lemma 5. Recall s_c is a permutation of s_1. In particular, s_c is sorted in ascending order of s_1, and we will assume that s_1 is sorted in ascending order of s_0. Since both vectors have the same elements, the only changes are the ones due to positional changes. If nobody changes positions $s_c = s_1$, and $XM = 0$, trivially. Hence, we will assume from now on that $\exists\, i \leq n$ such that $s_{i1} \neq s_{ic}$.

Denote the difference between s_1 and s_c by

$$\eta_i \equiv s_{i1} - s_{ic}.$$

Also, denote the set of individual indices by $I = \{1, \ldots, n\}$.

Since we want to establish that

$$\sum_{i \in I} (s_{i1} - s_{ic}) s_{i0} \leq 0,$$

we need only include in the sum those individuals who changed position, since $\eta_i = 0$ for those who did not change position.

The set of individual indices with non-zero positional changes is denoted by $\tilde{I} = \{1, \ldots, m\}$, where $m \leq n$, and the index 1 now corresponds to the individual with the lowest initial income, who changed positions across periods.

The next claim will be useful in what follows.

Claim

$$\sum_{i=0}^{k} \eta_{m-i} \leq 0 \quad \forall k < m.$$

The proof is as follows: Start with the income changer who was initially richest; this is person m. Note that $\eta_m = s_{m1} - s_{mc} < 0$, since this person must have moved lower in the distribution.

Now consider the two initially richest income changers, indexed by m and m-1. Add their η terms

$$\eta_m + \eta_{m-1} = (s_{m1} + s_{m-1,1}) - (s_{mc} + s_{m-1,c}).$$

The terms in s_c are the two largest shares because s_c is ordered in ascending order of s_1. The terms in s_1 may or may not be the largest, hence

$$\eta_m + \eta_{m-1} \leq 0$$

Now continue to the top three, top four, etc. The same logic as before yields

$$\sum_{i=0}^{k} \eta_{m-i} = \sum_{i=0}^{k} s_{m-i,1} - \sum_{i=0}^{k} s_{m-i,c}.$$

Again, note that the elements in s_c sum are the largest k+1 final shares, while the elements in the s_1 sum need not be the largest k+1 final shares. This establishes the claim.

Since $\forall i \in \tilde{I}$, $\eta_i \neq 0$, we can partition the index set $\tilde{I}$ into alternating subsets of contiguous indices with positive and negative changes. That is, we can express

$$\tilde{I} = \{M_1, M_2, \ldots, M_h\} \quad h \leq m,$$

where the subsets of the partitions M_k have the following properties:

i) For all $i \in M_k$, it is either true that $\eta_i > 0$ or $\eta_i < 0$.

ii) For any sets M_k and M_l, with $k < l$, and for all $i \in M_k$ and $j \in M_l$, we have $i < l$.

Note that in the partition $\tilde{I} = \{M_1, M_2, \ldots, M_h\}$, the first subset, M_1, contains observations with positive changes. This is because the index 1 corresponds to the initially poorest individual changer. This changer had to move up, because had he moved down, somebody poorer than he would have had to have moved up, in which case this changer would not have been the initially poorest. A similar logic establishes that the last subset in the partition, M_h, contains elements with negative changes η_i.

To simplify notation we will denote the subsets with positive elements by P_j and the ones with negative changes by N_j. Hence, we can reexpress our partition as

$$\tilde{I} = \{P_1, N_1, \ldots, P_g, N_g\} \quad g \leq m,$$

Furthermore, for each of these P_j subsets denote their maximum elements as $\hat{p}_j = \max P_j$.

Next, define the following sums over such subsets:

$$SP_j = \sum_{i \in P_j} \eta_i; \quad SN_j = \sum_{i \in N_j} \eta_i; \quad S_j = SP_j + SN_j$$
$$XP_j = \sum_{i \in P_j} \eta_i s_{i0}; \quad XN_j = \sum_{i \in N_j} \eta_i s_{i0}; \quad X_j = XP_j + XN_j.$$

Consider first the sum

$$\begin{aligned}
X_g &= XN_g + XP_g \\
&= \sum_{i \in N_g} \eta_i s_{i0} + \sum_{i \in P_g} \eta_i s_{i0} \\
&\leq \sum_{i \in N_g} \eta_i s_{i0} + s_{\hat{p}_g 0} \sum_{i \in P_g} \eta_i \quad (\text{since } \hat{p}_g = \max P_g) \\
&= \sum_{i \in N_g} \eta_i s_{i0} + s_{\hat{p}_g 0}(S_g - SN_g) \quad (\text{by definition of } SP_g) \\
&= \sum_{i \in N_g} \eta_i (s_{i0} - s_{\hat{p}_g 0}) + s_{\hat{p}_g 0} S_g \quad (\text{by definition of } SN_g).
\end{aligned}$$

Observe that $\forall i \in N_g$, i) $\eta_i < 0$, ii) $s_{i0} - s_{\hat{p}_g 0} \geq 0$ (by the fact that for any k if $i \in N_k$ and $j \in P_k$, then $j < i$), and $S_g \leq 0$, by the above Claim. Therefore $X_g \leq 0$.

Now, consider adding the next pair of subsets, XN_{g-1} and XP_{g-1}, to X_g

$$\begin{aligned}
X_g + X_{g-1} &= X_g + XN_{g-1} + XP_{g-1} \\
&\leq \sum_{i \in N_g} \eta_i (s_{i0} - s_{\hat{p}_g 0}) + s_{\hat{p}_g 0} S_g + \sum_{i \in N_{g-1}} \eta_i s_{i0} + \sum_{i \in P_{g-1}} \eta_i s_{i0} \\
&\leq \sum_{i \in N_g} \eta_i (s_{i0} - s_{\hat{p}_g 0}) + s_{\hat{p}_g 0} S_g + \sum_{i \in N_{g-1}} \eta_i s_{i0} + \\
&\cdots + s_{\hat{p}_{g-1},0} \sum_{i \in P_{g-1}} \eta_i \qquad (\text{since } \hat{p}_{g-1} = \max P_{g-1}) \\
&= \sum_{i \in N_g} \eta_i (s_{i0} - s_{\hat{p}_g 0}) + s_{\hat{p}_g 0} S_g + \sum_{i \in N_{g-1}} \eta_i s_{i0} + \\
&\cdots + s_{\hat{p}_{g-1},0}(S_{g-1} - SN_{g-1}) \quad (\text{since } S_{g-1} = SP_{g-1} + SN_{g-1}) \\
&= \sum_{i \in N_g} \eta_i (s_{i0} - s_{\hat{p}_g 0}) + s_{\hat{p}_g 0} S_g + \sum_{i \in N_{g-1}} \eta_i (s_{i0} - s_{\hat{p}_{g-1},0}) + \\
&\cdots + s_{\hat{p}_{g-1},0} S_{g-1} \quad (\text{by definition of } SN_{g-1}).
\end{aligned}$$

In this last expression we can establish that the summations over N_g and N_{g-1} are negative since

$$\eta_i < 0, \quad s_{i0} \geq s_{\hat{p}_{g-1},0} \quad \forall i \in N_{g-1}$$

$$\eta_i < 0, \quad s_{i0} \geq s_{\hat{p}_g 0} \quad \forall i \in N_g$$

Furthermore, $s_{\hat{p}_g 0} S_g + s_{\hat{p}_{g-1},0} S_{g-1} \leq 0$, since both $S_g \leq 0$ and $S_g + S_{g-1} \leq 0$ (by the above Claim), and $s_{\hat{p}_g 0} > s_{\hat{p}_{g-1},0}$ by construction. In other words, we have established that $X_g + X_{g-1} \leq 0$.

From the above arguments, it is clear that by following the previous steps a finite number of times we can establish that

$$\sum_{j=1}^{g} X_j \leq 0$$

but $\sum_{j=1}^{g} X_j = XM$, so this completes the proof of the Lemma. $\square$

Proposition 7 ***Convergence in Shares and Lorenz Dominance***

i) A Lorenz-improvement ($LC_1 \succ LC_0$) implies share convergence ($\delta_s < 0$).

ii) A Lorenz-worsening is consistent with either convergent or divergent share changes.

Proof of Proposition 7. Consider the share change regression (6)

$$\Delta s \equiv s_1 - s_0 = \gamma_s + \delta_s s_0 + u_s$$

The coefficient δ_s equals

$$\delta_s = \frac{cov(\Delta s, s_0)}{V(s_0)}.$$

Hence, its sign will be determined by the sign of the covariance

$$\begin{aligned} cov(\Delta s, s_0) &= n^{-1} \sum_i (s_{i1} - s_{i0}) s_{i0} - \overline{\Delta s} \cdot \overline{s_0} \\ &= n^{-1} \sum_i (s_{i1} - s_{i0}) s_{i0} \qquad \text{(since the average share-change is zero)} \\ &= n^{-1} \sum_i [(s_{i1} - s_{ic}) + (s_{ic} - s_{i0})] s_{i0} \\ &= XM + SM \end{aligned}$$

for XM and SM defined in (9). Hence,

$$sign(\delta_s) = sign(XM + SM).$$

By Lemma 4, a Lorenz-improvement $LC_1 \succ LC_0 \implies SM < 0$. By Lemma 5, $XM \leq 0$ always. Hence, if $LC_1 \succ LC_0$ then $XM + SM < 0$, and therefore $\delta_s < 0$. This proves part i) of Proposition 7.

Finally, part ii) is established by noting that whether share changes are convergent or divergent depends on the sign of $cov(\Delta s, s_0)$, which equals $SM + XM$. If we have positional changes and Lorenz-worsening, then $SM > 0$ and $XM < 0$. Therefore, the sign of the sum depends on the magnitudes of the components. Either component can be larger in absolute value than the other, and thus both convergence and divergence in shares are possible.

In particular, consider the transition

$$[2, 4, 6] \rightarrow [85, 8, 7]$$

is an example of positional changes, a Lorenz-worsening, and convergent share changes, i.e. $\delta_s < 0$. In contrast, the transition

$$[2, 4, 6] \rightarrow [8, 7, 85]$$

is an example of positional changes, a Lorenz-worsening, and divergent share changes, i.e. $\delta_s > 0$. □

Lemma 6 *Let μ_t denote the mean income in period t, β_d and β_s denote the convergence coefficients given by regressions (10) and (11) in dollars and in shares, respectively, and g denote the economy-wide growth rate in incomes between year 0 and year 1. Then*

$$\beta_d = \beta_s \frac{\mu_1}{\mu_0} = \beta_s(1 + g).$$

Proof of Lemma 6. The regression in dollars (10) is

$$d_1 = \alpha_d + \beta_d d_0 + u_d.$$

Dividing this equation by μ_1 we obtain

$$\begin{aligned} s_1 &= \frac{\alpha_d}{\mu_1} + \beta_d \frac{d_0}{\mu_1} + \frac{u_d}{\mu_1} \\ &= \frac{\alpha_d}{\mu_1} + \beta_d \frac{d_0}{\mu_0} \frac{\mu_0}{\mu_1} + \frac{u_d}{\mu_1} \\ &= \frac{\alpha_d}{\mu_1} + \beta_d s_0 \frac{\mu_0}{\mu_1} + u_s. \end{aligned}$$

Hence,

$$\alpha_s = \frac{\alpha_d}{\mu_1}; \beta_s = \beta_d \frac{\mu_0}{\mu_1} = \beta_d \frac{1}{1+g}.$$

The Lemma follows from this last equation.

□

Proposition 8 ***Convergence in Dollars and Lorenz Dominance***
Let δ_d be defined by the change-on-initial regression (7) when income is measured in dollars, and let g denote the economy-wide growth rate in incomes between year 0 and year 1.

i) If $g \leq 0$, a Lorenz-improvement ($LC_1 \succ LC_0$) implies convergence in dollars ($\delta_d < 0$).

ii) A Lorenz-worsening ($LC_1 \prec LC_0$) is compatible with both convergence and divergence in dollars.

Proof of Proposition 8. Proof of part i):

By Proposition 7.i) a Lorenz-improvement ($LC_1 \succ LC_0$) implies share convergence, $\delta_s < 0$, (or $\beta_s < 1$). Coupling this with Lemma 6, which establishes that $\beta_d = \beta_s(1+g)$, it follows that whenever $g \leq 0$, a Lorenz-improvement implies $\beta_d = \beta_s(1+g) < 1$, i.e. we have convergence in dollars.

Proof of part ii):

By Proposition 7.ii), a Lorenz-worsening ($LC_0 \succ LC_1$) is consistent with either $\beta_s > 1$ or $\beta_s < 1$. Therefore from Lemma 6 it follows that β_d can be smaller or larger than one, and thus Lorenz-worsening is consistent with both dollar convergence and dollar divergence.

As a form of illustration, the transition

$$[1, 3] \rightarrow [5, 1]$$

is an example of a Lorenz-worsening, and convergent dollar changes, i.e. $\delta_d < 0$. In contrast, the transition

$$[1, 3] \rightarrow [1, 5]$$

is an example of a Lorenz-worsening, and divergent dollar changes, i.e. $\delta_d > 0$. $\square$

Proposition 9 ***Log-income Panel Changes and Lorenz Dominance under a Single Rank-Preserving Transfer Sufficiently High Up in the Income Distribution***

Let gm denote the geometric mean of income at period 0, and $\exp(1) = 2.718$. *Consider two individuals i and j such that $d_{i0} > d_{j0} > gm * \exp(1)$. Let $h > 0$ be a sufficiently small rank-preserving transfer between i and j.*

a) If such a transfer h is equalizing, it produces a Lorenz-improvement $LC_1 \succ LC_0$, rising inequality as gauged by the log-variance ($V(\ln d_1) > V(\ln d_0)$), and a divergent regression coefficient ($\delta_{log} > 0$).

b) If such a transfer h is disequalizing, it produces a Lorenz-worsening $LC_1 \prec LC_0$, falling inequality as gauged by the log-variance ($V(\ln d_1) < V(\ln d_0)$), and a convergent regression coefficient ($\delta_{log} < 0$).

Proof of Proposition 9. Let gm denote the geometric mean of incomes at period 0, i.e.

$$gm = exp\left(n^{-1} \sum_i \ln d_i\right).$$

Let $h > 0$ be a sufficiently small rank-preserving transfer. Consider two individuals i and j such that $d_{i0} > d_{j0} > gm * exp(1)$ and assume that the income change when going from period 0 to 1 is the transfer h between i and j, all other incomes remaining unchanged.

It follows from Fields and Fei (1978) that if the transfer is equalizing it will lead to a Lorenz-improvement, and the opposite will occur if the transfer is disequalizing. Similarly, Cowell (2011) establishes that under the stated assumptions $V(\ln d)$ will change in the directions established by the Proposition. The only result remaining to establish is the sign of the coefficient $\delta_{\log}$ in a log-change regression (12) under the stated conditions.

Consider the case a) of a single rank-preserving *equalizing* transfer. That is the transfer goes from the richer person i to the poorer person j. Under the stated assumptions the sign of $\delta_{\log}$ will be determined by the covariance

$$cov(\Delta \ln d, \ln d_0) = n^{-1} \sum_l (\ln d_{l1} - \ln d_{l0}) \ln d_{l0} - \overline{\Delta \ln d} \cdot \overline{\ln d_0}$$

Note that all terms in the summation are zero except for $l \in \{i, j\}$, so we have

$$\begin{aligned} cov(\Delta \ln d, \ln d_0) &= n^{-1} \left[(\Delta \ln d_i) \ln d_{i0} + (\Delta \ln d_j) \ln d_{j0}\right] - \overline{\Delta \ln d} \cdot \overline{\ln d_0} \\ &= n^{-1} \left[(\ln(d_{i0} - h) - \ln d_{i0}) \ln d_{i0}\right] + \cdots \\ &\cdots + n^{-1} \left[(\ln(d_{j0} + h) - \ln d_{j0}) \ln d_{j0}\right] - \overline{\Delta \ln d} \cdot \overline{\ln d_0} \end{aligned}$$

A First-order Taylor expansion around $h = 0$ for the first two terms is

$$\begin{aligned} n^{-1} \left[(\ln(d_{i0} - h) - \ln d_{i0}) \ln d_{i0}\right] &\cong -\frac{\ln d_{i0}}{d_{i0}} \frac{h}{n} \\ n^{-1} \left[(\ln(d_{j0} + h) - \ln d_{j0}) \ln d_{j0}\right] &\cong \frac{\ln d_{j0}}{d_{j0}} \frac{h}{n}. \end{aligned}$$

A similar expansion for the average log-income change is

$$\overline{\Delta \ln d} \cong \frac{h}{n} \left(\frac{1}{d_{j0}} - \frac{1}{d_{i0}} \right).$$

Hence, for a marginal transfer h

$$cov(\Delta \ln d, \ln d_0) \cong \frac{h}{n} \left(\frac{\ln d_{j0} - \overline{\ln d_0}}{d_{j0}} - \frac{\ln d_{i0} - \overline{\ln d_0}}{d_{i0}} \right).$$

The sign of this covariance will be determined by the behavior of the function

$$\frac{\ln x - \overline{\ln d_0}}{x}$$

with derivative

$$\frac{1 - \ln x + \overline{\ln d_0}}{x^2}.$$

This derivative will be negative when

$$x > exp(1) * gm.$$

Hence, if individuals have incomes $d_{i0} > d_{j0} > exp(1) * gm$, and an equalizing transfer is made from i to j, then

$$\frac{\ln d_{j0} - \overline{\ln d_0}}{d_{j0}} - \frac{\ln d_{i0} - \overline{\ln d_0}}{d_{i0}}$$

will have a positive sign, and so $\delta_{\log} > 0$. The case of a disequalizing transfer is proved similarly.

□

Lemma 7 *Let PSM be given by equation (13), then:*

i) A Lorenz-improvement ($LC_1 \succ LC_0$) *implies* $PSM > 0$.

ii) A Lorenz-worsening ($LC_1 \prec LC_0$) *implies* $PSM < 0$.

Proof of Lemma 7. As in the proof of Lemma 4, when there is a Lorenz-improvement we can go from s_0 to s_c through a sequence of rank-preserving equalizing transfers $h(j, i)$. For any such transfer $h(j, i)$ it is the case that:

i) they appear once with a positive sign and once with a negative sign, and

ii) at any given stage of the sequence of transfers, the sender j is always richer than the receiver i.

Hence, for each transfer $h(j, i)$, we have that the product

$$h(j, i) \left(\frac{1}{\tilde{s}_i} - \frac{1}{\tilde{s}_j} \right) \geq 0,$$

where $\tilde{s}_i$ and $\tilde{s}_j$ are the shares of individuals i and j, respectively, at the given stage of the sequence of transfers where $h(i, j)$ takes place.

Since each of these transfers are rank-preserving, it follows that at any given stage of the sequence of transfers, $\tilde{s}_j - \tilde{s}_i \geq 0$ implies

$$h(j,i)\left(\frac{1}{s_{i0}} - \frac{1}{s_{j0}}\right) \geq 0.$$

This in turn implies that

$$\begin{aligned} PSM &= \frac{1}{n}\sum_i \frac{s_{ic} - s_{i0}}{s_{i0}} \\ &= \frac{1}{n}\sum_i \frac{\sum_{j=1}^n h(j,i) - \sum_{j=1}^n h(i,j)}{s_{i0}} \end{aligned}$$

will be the sum of terms $h(j,i)(\frac{1}{s_{i0}} - \frac{1}{s_{j0}})$ for all transfers $h(j,i)$. Since all these terms are non-negative, and some will be strictly positive, then the average percentage change in shares will be positive.

□

Lemma 8

$$PXM = \frac{1}{n}\sum_i \frac{s_{i1} - s_{ic}}{s_{i0}} \geq 0.$$

Proof of Lemma 8. Recall s_c is a permutation of s_1. In particular, s_c is sorted in ascending order of s_1, and as in the proof of Lemma 5 we will assume that s_1 is sorted in ascending order of s_0. Since both vectors have the same elements, the only changes are the ones due to positional changes. If nobody changes positions $s_c = s_1$, and $PXM = 0$, trivially. Hence, we will assume from now on that $\exists\, i \leq n$ such that $s_{i1} \neq s_{ic}$.

As before, denote the difference between s_1 and s_c by

$$\eta_i \equiv s_{i1} - s_{ic}.$$

Also, denote the set of individual indices by $I = \{1, \ldots, n\}$.

Since we want to establish that

$$\sum_{i \in I} \frac{s_{i1} - s_{ic}}{s_{i0}} \geq 0,$$

we need only include in the sum those individuals who changed position, since $\eta_i = 0$ for those who did not change position.

The set of individual indices with non-zero positional changes is denoted by $\tilde{I} = \{1, \ldots, m\}$, where $m \leq n$, and the index 1 now corresponds to the individual with the lowest initial income, who changed positions across periods.

The next claim will be useful in what follows.

Claim

$$\sum_{i=1}^{k} \eta_i \geq 0 \quad \forall k < m.$$

This claim bears a close parallel to a claim made in the proof of Lemma 5. The difference between these two claims is that in the claim from Lemma 5 the cumulative sum of η_i is made from the largest elements to the smallest ones, while in this claim the sum is made in the opposite direction. The proof of this claim is established as follows: Start with the income changer who was initially poorest; this is person 1. Note that $\eta_1 = s_{11} - s_{1c} > 0$, since this person must have moved up in the distribution.

Now consider the two initially poorest income changers, indexed by 1 and 2. Add their η terms

$$\eta_1 + \eta_2 = (s_{11} + s_{21}) - (s_{1c} + s_{2c}).$$

The terms in s_c are the two smallest shares because s_c is ordered in ascending order of s_1. The terms in s_1 may or may not be the smallest, hence

$$\eta_1 + \eta_2 \geq 0$$

Now continue to the bottom three, top four, etc. The same logic as before yields

$$\sum_{i=1}^{k} \eta_i = \sum_{i=1}^{k} s_{i1} - \sum_{i=1}^{k} s_{ic}.$$

Again, note that the elements in s_c sum are the smallest k final shares, while the elements in the s_1 sum need not be the smallest k final shares. This establishes the claim.

As in the proof of Lemma 5 we can partition the index set $\tilde{I}$ into g alternating subsets of contiguous indices with positive and negative changes.

That is, we can express

$$\tilde{I} = \{P_1, N_1, \ldots, P_g, N_g\} \quad g \leq m.$$

Furthermore, for each of these P_j subsets denote their maximum elements as $\check{n}_j = \min N_j$.

Next, define the following sums over such subsets:

$$SP_j = \sum_{i \in P_j} \eta_i; \quad SN_j = \sum_{i \in N_j} \eta_i; \quad S_j = SP_j + SN_j$$
$$PXP_j = \sum_{i \in P_j} \frac{\eta_i}{s_{i0}}; \quad PXN_j = \sum_{i \in N_j} \frac{\eta_i}{s_{i0}}; \quad PX_j = PXP_j + PXN_j.$$

Consider first the sum

$$\begin{aligned}
PX_1 &= PXN_1 + PXP_1 \\
&= \sum_{i \in N_1} \frac{\eta_i}{s_{i0}} + \sum_{i \in P_1} \frac{\eta_i}{s_{i0}} \\
&\geq \sum_{i \in P_1} \frac{\eta_i}{s_{i0}} + \frac{1}{s_{\check{n}_1 0}} \sum_{i \in N_1} \eta_i \quad \text{(since } \check{n}_1 = \min N_1 \text{ and } \eta_i < 0 \; \forall i \in N_1\text{)} \\
&= \sum_{i \in P_1} \frac{\eta_i}{s_{i0}} + \frac{1}{s_{\check{n}_1 0}} (S_1 - SP_1) \quad \text{(by definition of } SN_1\text{)} \\
&= \sum_{i \in P_1} \eta_i \left(\frac{1}{s_{i0}} - \frac{1}{s_{\check{n}_1 0}} \right) + \frac{1}{s_{\check{n}_1 0}} S_1 \quad \text{(by definition of } SP_1\text{)}.
\end{aligned}$$

Observe that i) $\eta_i > 0$, ii) $s_{i0} - s_{\check{n}_1 0} \leq 0 \;\; \forall i \in P_1$, and $S_1 \geq 0$, by the above Claim. Therefore $PX_1 \geq 0$.

Now, consider adding the next pair of subsets, PXN_2 and PXP_2, to PX_1,

$$\begin{aligned}
PX_1 + PX_2 &= PX_1 + PXN_2 + PXP_2 \\
&\geq \sum_{i\in P_1} \eta_i \left(\frac{1}{s_{i0}} - \frac{1}{s_{\check{n}_1 0}}\right) + \frac{1}{s_{\check{n}_1 0}} S_1 + \sum_{i\in N_2} \frac{\eta_i}{s_{i0}} + \sum_{i\in P_2} \frac{\eta_i}{s_{i0}} \\
&\geq \sum_{i\in P_1} \eta_i \left(\frac{1}{s_{i0}} - \frac{1}{s_{\check{n}_1 0}}\right) + \frac{1}{s_{\check{n}_1 0}} S_1 + \frac{1}{s_{\check{n}_2 0}} \sum_{i\in N_2} \eta_i + \\
&\cdots + \sum_{i\in P_2} \frac{\eta_i}{s_{i0}} \qquad \text{(since } \check{n}_2 = \min N_2 \text{ and } \eta_i < 0 \ \forall i \in N_2\text{)} \\
&= \sum_{i\in P_1} \eta_i \left(\frac{1}{s_{i0}} - \frac{1}{s_{\check{n}_1 0}}\right) + \frac{1}{s_{\check{n}_1 0}} S_1 + \frac{1}{s_{\check{n}_2 0}} (S_2 - SP_1) + \\
&\cdots + \sum_{i\in P_2} \frac{\eta_i}{s_{i0}} \qquad \text{(since } S_2 = SN_2 + SP_2\text{)} \\
&= \sum_{i\in P_1} \eta_i \left(\frac{1}{s_{i0}} - \frac{1}{s_{\check{n}_1 0}}\right) + \frac{1}{s_{\check{n}_1 0}} S_1 + \frac{1}{s_{\check{n}_2 0}} S_2 + \\
&\cdots + \sum_{i\in P_2} \eta_i \left(\frac{1}{s_{i0}} - \frac{1}{s_{\check{n}_2 0}}\right) \quad \text{(by definition of } SP_2\text{).}
\end{aligned}$$

From this last expression we can establish that the summations over P_1 and P_2 are positive since

$$\eta_i > 0, \quad s_{i0} \leq s_{\check{n}_2 0} \quad \forall i \in P_2$$

$$\eta_i > 0, \quad s_{i0} \leq s_{\check{n}_1 0} \quad \forall i \in P_1.$$

Furthermore,

$$\frac{1}{s_{\check{n}_1 0}} S_1 + \frac{1}{s_{\check{n}_2 0}} S_2 \geq 0,$$

since both $S_1 \geq 0$ and $S_1 + S_2 \geq 0$ (by the above Claim) and also $s_{\check{n}_1 0} \leq s_{\check{n}_2 0}$, by construction. In other words, we have established that $PX_1 + PX_2 \geq 0$.

From the above arguments, it is clear that by following the previous steps a finite number of times we can establish that

$$\sum_{j=1}^{g} PX_j \geq 0$$

but $\sum_{j=1}^{g} PX_j = PXM$, so this completes the proof of the Lemma. $\square$

Proposition 10 ***Convergence in Exact Proportional Changes and Lorenz Dominance***

Let θ be defined by the exact proportional change regression (5).

i) A Lorenz-improvement ($LC_1 \succ LC_0$) implies convergence in exact proportional changes ($\theta < 0$).

ii) A Lorenz-worsening ($LC_1 \prec LC_0$) is compatible with both convergent and divergent exact proportional changes.

Proof of Proposition 10. Note first that

$$\begin{aligned} E\left(\frac{\Delta s}{s_0}\right) &= \frac{1}{n}\sum_i \frac{s_{i1} - s_{i0}}{s_{i0}} \\ &= \frac{1}{n}\sum_i \frac{(s_{i1} - s_{ic}) + (s_{ic} - s_{i0})}{s_{i0}} \\ &= PXM + PSM \end{aligned}$$

From Lemma 7 whenever there is a Lorenz-improvement $PSM > 0$, and by Lemma 8 $PXM \geq 0$ always, hence the average percentage change in shares will be positive.

Now, recall from Lemma 2 that whenever average percentage changes in shares are positive, the exact proportional changes are convergent (and vice versa), i.e.

$$0 < E\left(\frac{s_1 - s_0}{s_0}\right) \iff \theta < 0.$$

This establishes part i) of the Proposition.

Finally, part ii) is established by noting that in the case of a Lorenz-worsening ($LC_1 \prec LC_0$), PSM will have an opposite sign to PXM. Therefore, $E\left((s_1 - s_0)/s_0\right)$, and thus θ, can have any sign.

In particular, consider the transition

$$[2, 4, 6] \to [85, 8, 7]$$

is an example of positional changes, a Lorenz-worsening, and convergent exact proportional changes, i.e. $\theta < 0$. In contrast, the transition

$$[2, 4, 6] \to [8, 7, 85]$$

is an example of positional changes, a Lorenz-worsening, and divergent exact proportional changes, i.e. $\theta > 0$.

$\square$

Proposition 11 ***Convergence in Shares and Changes in Transfer Sensitive Inequality Indices under Single Lorenz-Crossing from Above***

If the Lorenz curve of d_1 intersects that of d_0 once from above and $CV(d_1) \leq CV(d_0)$, then all measures in the $I_{TS}(d)$ class and the coefficients of the linear regressions of share changes (δ_s) are linked as follows:

i) $I_{TS}(d_1) < I_{TS}(d_0)$

ii) $\delta_s < 0$.

Proof of Proposition 11. As previously mentioned, part i) is derived in Shorrocks and Foster (1987), Corollary 1.

Lemma 1 and Proposition 1 applied to shares imply $\delta_s < 0$, whenever $CV_1 < CV_0$.

It only remains to show that if $CV_1 = CV_0$ it will be the case that $\delta_s < 0$. In the proof of Proposition 1 we established that

$$\Delta V(s) = \delta_s(\delta_s + 2)V(s_0) + V(u_s).$$

Hence, if the $\Delta CV = 0$, this means that

$$0 = \delta_s(\delta_s + 2)V(s_0) + V(u_s),$$

which in turn implies that $\delta_s \leq 0$.

If $V(u_s) > 0$, it must be the case that $\delta_s < 0$. If $V(u_s) = 0$ then $\delta_s = 0$ and $u_i = 0$ for all individuals. In other words, the regression line would be

$$\begin{aligned}\Delta s_i =& \gamma_s + \delta_s s_{i0} + u_i \\ \Delta s_i =& \gamma_s,\end{aligned}$$

yet since $E(\Delta s) = 0$, it is implied that $\gamma_s = 0$. In other words, if $V(u_s) = 0$ then $s_{i1} = s_{i0}$ for all individuals. Yet this is a contradiction, since we assumed a changing Lorenz curve across periods. In summary, the assumption $V(u_s) = 0$ is false, and we are left with the case where $\delta_s < 0$. $\square$