Long Term Growth Model (LTGM v4.3) - Model Description

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- **NEW** in LTGM v4: effect of growth on poverty (via log-normal income distribution, Section 5)
- The neoclassical growth model is based on Solow (1956), Swan (1956) and Hevia and Loayza (2012)
  - There are only two key parts: the production function and capital accumulation.
- Model 1: assume a path for the investment share of GDP \((I/Y)\) → implied per-capita GDP growth.
- Model 2: assume a path of growth in GDP per capita → required investment share of GDP \((I/Y)\).
- Model 3: assume a path for the savings share of GDP \((S/Y)\) \(→\) implied per-capita GDP growth.
- A Current Account Balance or External Debt constraint converts savings \((S/Y)\) into \((I/Y)\) in Model 3. The constraint also allows savings to be calculated as residual in Model 1 and 2 (see Section 3 for details).
- Section 4 summarizes the drivers of per-capita GDP growth in one equation (and compares to the ICOR).

1 Model 1: Growth given investment

1.1 The production function

I assume a standard production function where \(Y_t\) is GDP, \(A_t\) is the total factor productivity, \(K_t\) is the capital stock, and \(h_tL_t\) is effective labor used in production, which can be further decomposed as \(h_t\) human capital per worker, and \(L_t\) is the number of workers. \(\beta\) is the labor share.

\[
Y_t = A_t K_t^{1-\beta} (h_t L_t)^\beta
\]  

(1)

\(N_t\) is the total population and so the number of workers can be decomposed into \(L_t = \varrho_t \omega_t N_t\) where \(\varrho_t\) is the participation rate (labor force/working age population) and \(\omega_t\) is the working age population-to-total population ratio.\[1\]

We can divide by \(N_t\) or \(L_t\) to get all variables in per capita or per worker terms. In terms of notation, the default is per worker, but we add “PC” to denote per capita (i.e. \(y_t\) is output per worker and \(y_t^{PC}\) is output per capita). \(k_t\) is capital per worker. \(h_t\) is already in per worker terms terms.

\[
y_t^{PC} = \frac{Y_t}{N_t} = \frac{Y_t}{L_t} \varrho_t \omega_t = y_t \varrho_t \omega_t
\]

\[y_t = \frac{Y_t}{L_t} = A_t k_t^{1-\beta} h_t^\beta
\]

\[y_t^{PC} = \varrho_t \omega_t y_t = A_t \varrho_t \omega_t k_t^{1-\beta} h_t^\beta
\]

From this equation we can calculate growth rates from \(t\) to \(t+1:\)

\[
y_{t+1} \quad \quad y_t
\]

\[
= \left[ \frac{A_t k_{t+1}^{1-\beta} h_{t+1}^{\beta}}{A_t k_t^{1-\beta} h_t^\beta} \right] \quad = \left[ \frac{k_{t+1}}{k_t} \right]^{1-\beta} \left[ \frac{h_{t+1}}{h_t} \right]^\beta
\]

Rewrite this equation in terms of \(g_{y,t+1}\), the growth rate of GDP per worker from \(t\) to \(t+1\) (eg 0.05).

\[1\]We assume no unemployment or underemployment, so everyone in the labor force works. In an earlier version of this note (with different notation), I assumed that \(\varrho_t = \omega_t = 1\) so that population and the labor force were the same.
\[ 1 + g_{t+1} = (1 + g_{A,t+1}) [1 + g_{k,t+1}]^{1-\beta} [1 + g_{h,t+1}]^{3} \]  

(2)

In Equation (2), output growth is driven by productivity growth \((g_{A,t+1})\), capital deepening \((g_{k,t+1})\), and human capital per worker. The next step is to write capital accumulation in terms of growth rates and in per worker terms. Start with Equation 4 and divide everything by \(L_t\).

\[
\begin{bmatrix} K_{t+1} \\ L_{t+1} \end{bmatrix} = \begin{bmatrix} \varphi_{t+1} \\ \omega_{t+1} \end{bmatrix} \begin{bmatrix} \varphi_t \\ \omega_t \end{bmatrix} \begin{bmatrix} K_t \\ L_t \end{bmatrix} \]

Now write in terms of growth rates and in per worker terms.

\[
k_{t+1} \{1 + g_{k,t+1}\} \{1 + g_{w,t+1}\} (1 + g_{N,t+1}) = (1 - \delta) k_t + i_t
\]

We can divide everything by \(k_t\)

\[
\frac{k_{t+1}}{k_t} (1 + g_{N,t+1}) \{1 + g_{o,t+1}\} \{1 + g_{w,t+1}\} = (1 - \delta) + \frac{i_t}{k_t} y_t
\]

Then divide and multiply by \(y_t\) (output per worker). Where \(\frac{I_t}{Y_t}\) is the investment share of GDP and \(\frac{K_t}{Y_t}\) is the capital to output ratio.

\[
(1 + g_{k,t+1})(1 + g_{N,t+1}) \{1 + g_{o,t+1}\} \{1 + g_{w,t+1}\} = (1 - \delta) + \frac{i_t}{y_t} \frac{y_t}{k_t}
\]

Rearrange to get the growth rate of capital per worker.\(^2\)

\[
(1 + g_{k,t+1}) = \frac{1}{1 + g_{N,t+1})(1 + g_{o,t+1})(1 + g_{w,t+1})} \left( (1 - \delta) + \frac{I_t}{Y_t} \frac{K_t}{Y_t} \right)
\]

(5)

Equation (5) determines capital deepening (in per worker terms).

To solve the model, we need \(K_t/Y_t\). We take the initial value \(K_0/Y_0\) from the MFMod Database, PWT 8.1 and 9. For later periods, we need to update the capital-to-output ratio in the next period using Equation 6 and the values for \(g_{k,t+1}\) and \(g_{y,t+1}\) have we calculated in Equation 5 and 2.

\[
\frac{K_{t+1}}{Y_{t+1}} = \frac{k_{t+1}}{y_{t+1}} = \frac{k_{t+1}}{y_{t+1}} \frac{y_t}{k_t} \frac{k_t}{y_t}
\]

\(^2\)Note that \(\frac{I_t}{Y_t} = \frac{i_t}{y_t}\) and \(\frac{K_t}{Y_t} = \frac{k_t}{y_t}\) because the \(L_t\) in numerator and denominator cancel.
\[
\frac{K_{t+1}}{Y_{t+1}} = \frac{(1 + g_{k,t+1}) K_t}{(1 + g_{y,t+1}) Y_t}
\]  

(6)

1.3 Steps to solve the model

In order to solve the model, we first need to input some exogenous variables:

- Parameters for \(\beta\) (the labor share), \(\delta\) (the capital depreciation rate), and \(\frac{K_0}{Y_0}\) (initial capital-to-output ratio)
- Assumption of paths for \(\left\{g_A,t, g_h,t, g_\omega,t, g_\varphi,t, g_N,t, \frac{I_t}{Y_t}\right\}_{t=1}^T\)

Then we can calculate the growth rate of GDP per capita using the following steps.

1. Calculate the growth rate of capital per worker using Equation 5 (using exogenous & predetermined variables)
2. Calculate the growth rate of output per worker using Equation 2 (using \(g_{k,t+1}\) from step 1)
3. Calculate the growth rate of output per capita using Equation 3 (using \(g_{y,t+1}\) from step 2)
4. Update next period’s capital to output ratio using Equation 6 (using \(g_{k,t+1}\) and \(g_{y,t+1}\) from steps 1 and 2)

1.4 Parameters and initial values

- Users can choose parameters and initial values from a range of data sources and time periods using the drop-down menus in the LTGM spreadsheet, or simply type in their own values.
  - Missing parameters or initial values are interpolated based on income group averages, which triggers a red warning cell.
- \(\beta\) (the labor share) is taken from PWT 8.1, 9.0, and 9.1 and the GTAP database. 0.4-0.7 are reasonable numbers.
- \(\delta\) (the capital depreciation rate) and \(\frac{K_0}{Y_0}\) (the initial capital-to-output ratio) are taken from PWT 8.1, 9.0, or 9.1.
- \(\frac{I_t}{Y_t}\) is the investment share of GDP. The user will have to make an assumption about the future path of \(\frac{I_t}{Y_t}\).
- \(g_{A,t+1}\) is exogenous total factor productivity growth, and reasonable numbers are 0% (pessimistic), 1% (moderate) or 2% (optimistic). Faster productivity growth can be obtained from (for example) technology adoption, greater competition, reduced regulation, or factors moving from less efficient to more efficient sectors. Historical averages calculated from PWT 8.1. 9.0 and 9.1 (or applying PWT 8.1 methodology with GTAP labor shares)
- \(g_{h,t+1}\) is the growth rate of labor participation. Historical data from the ILO or country authorities. In practice, the most important determinant of \(g_{h,t+1}\) is female labor force participation.
- \(g_{N,t+1}\) is exogenous population growth, and \(g_{\omega,t+1}\) is growth in the working age to total population ratio. These are both taken from the World Bank Health Nutrition and Population Statistics: Population estimates and projections (link), with \(g_{\omega,t+1}\) determined by the age structure of the population.
- \(g_{\psi,t+1}\) is exogenous human capital per worker growth and taken from PWT 8.1, 9.0 or 9.1. Higher human capital growth, eg from more schooling or more effective schooling, will increase growth.

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3The \(K_0/Y_0\) ratio is calculated as rkm/gdpna in PWT 8.1, 9.0 and rmu/gdpna in 9.1 (national accounts prices).
4Historical averages are taken from Fixed Domestic Investment as a share of Real GDP from MFMod (NEGDFTOTK/ /NYGPMK-TPKD) and WDI.
5Hevia and Loayza [2012] refer to this as \(h_t = e^{\phi(E_t)}\), which is the efficiency of a workers with \(E_t\) average years of schooling. If \(E_t = 0\), \(h_t = 1\), so \(h_t\) represents the efficiency of a worker with \(E_t\) years of education relative to one with no education. i.e. if \(h_t = 2\), for a country average years of schooling is twice as productive as a worker with no education. \(\phi(E)\) governs the return to an extra year of schooling, and in PWT is piecewise linear, where the marginal return to schooling is 13.4% for the first 4 years, 10.1% for following 4-8 years and 6.8% for years of schooling after that, schooling from Barro and Lee’s dataset v1.3 (Inklaar and Timmer 2013, see Equation 15 and 16).
2 Model 2: Calculating the Investment Share of GDP to achieve a given rate of GDP per capita growth

It is straightforward to rearrange the equations above to calculate the investment rate necessary to generate a required rate of per-capita GDP growth (users can also set a poverty target, see Section 5).

First, calculate the growth rate of output per worker consistent with desired rate of growth of output per capita using Equation 3

\[ 1 + g_{y,t+1} = \frac{1 + g_{pc}^{y,t+1}}{1 + g_{\omega,t+1}} \]

Next substitute this into Equation 2

\[
(1 + g_{A,t+1})[1 + g_{h,t+1}]^{1-\beta} [1 + g_{\omega,t+1}]^\beta = \frac{1 + g_{pc}^{y,t+1}}{1 + g_{\omega,t+1}}
\]

Next substitute Equation 5 into Equation 2 to remove the capital growth rate.

\[
(1 + g_{A,t+1})[1 + g_{h,t+1}]^{\beta} \left[ \frac{(1 - \delta) + \frac{I_t}{Y_t} \cdot \frac{K_t}{Y_t}}{(1 + g_{N,t+1})(1 + g_{\omega,t+1})(1 + g_{\theta,t+1})} \right]^{1-\beta} = \frac{1 + g_{pc}^{y,t+1}}{1 + g_{\omega,t+1}}
\]

Then do some algebra to isolate \( I/Y \) on the LHS.

\[
\left[ (1 - \delta) + \frac{I_t}{Y_t} \cdot \frac{K_t}{Y_t} \right]^{1-\beta} = \frac{\{1 + g_{pc}^{y,t+1}\} (1 + g_{N,t+1})^{1-\beta} (1 + g_{\omega,t+1})^{1-\beta} (1 + g_{\theta,t+1})^{1-\beta}}{(1 + g_{A,t+1}) [1 + g_{h,t+1}]^\beta [1 + g_{\omega,t+1}]^\beta [1 + g_{\theta,t+1}]^\beta}
\]

\[
\frac{I_t}{Y_t} \cdot \frac{K_t}{Y_t} = \left[ \frac{\{1 + g_{pc}^{y,t+1}\}^{1-\beta} (1 + g_{N,t+1})^{1-\beta} (1 + g_{\omega,t+1})^{1-\beta} (1 + g_{\theta,t+1})^{1-\beta}}{(1 + g_{A,t+1})^{1-\beta} [1 + g_{h,t+1}]^{1-\beta} [1 + g_{\omega,t+1}]^{1-\beta} [1 + g_{\theta,t+1}]^{1-\beta}} - (1 - \delta) \right]^{1-\beta}
\]

\[
\frac{I_t}{Y_t} = \frac{K_t}{Y_t} \left[ \frac{\{1 + g_{pc}^{y,t+1}\}^{1-\beta} (1 + g_{N,t+1})^{1-\beta} (1 + g_{\omega,t+1})^{1-\beta} (1 + g_{\theta,t+1})^{1-\beta}}{(1 + g_{A,t+1})^{1-\beta} [1 + g_{h,t+1}]^{1-\beta} [1 + g_{\omega,t+1}]^{1-\beta} [1 + g_{\theta,t+1}]^{1-\beta}} - (1 - \delta) \right]^{1-\beta}
\]

Given \( \{g_{pc}^{y,t}\}_{t=1} \) one can calculate required investment using Equation 7. As before, future values of \( K_t \) can be updated for period \( t+1, t+2.. \) using Equation 8 with the growth rate of capital calculated from Equation 5

Equation 7 states that required investment is increasing in the desired per capita growth rate \( g_{pc}^{y,t+1} \), the depreciation rate \( \delta \), the population growth rate \( g_{N,t+1} \) and also the capital-to-output ratio \( K_t/Y_t \). Growth in productivity \( (g_{A,t+1}) \), human capital \( (g_{h,t+1}) \), the working age population ratio \( (g_{\omega,t+1}) \) and the participation rate \( (g_{\theta,t+1}) \) all reduce the required investment rate.
3 The External Balance Constraint and Model 3 (Growth Given Savings)

Savings is converted into investment (and vice versa) using a binding external balance constraint which can either be in the form of a path for (i) \( \frac{CAB_t}{Y_t} \) or (ii) the external debt share of GDP \( \frac{D_t}{Y_t} \) (and foreign direct investment \( \frac{FDI_t}{Y_t} \)).

A Current Account Balance Constraint

The model is simplest assuming a path \( \frac{CAB_t}{Y_t} \):

\[
\frac{I_t}{Y_t} = \frac{S_t}{Y_t} - \frac{CAB_t}{Y_t}
\]

(8)

- **Model 3 (\( CAB/Y \) constraint):** Given an assumption for national savings as a share of GDP \( (S_t/Y_t) \), simply combine with the \( CAB_t/Y_t \) constraint and use Equation (8) to calculate \( I_t/Y_t \). Then use Step 1-4 from Model 1 (Section 1.3) to calculate growth.

- **Model 1 and 2 (\( CAB/Y \) constraint):** Rearrange (8) and combine the \( CAB_t/Y_t \) constraint with path for \( I_t/Y_t \) assumed (Model 1) or implied (Model 2) to generate implied savings as: \( S_t/Y_t = I_t/Y_t - CAB_t/Y_t \).

An External Debt Constraint

Alternatively, one can assume an external debt constraint, combined with a path for foreign direct investment \( (FDI_t) \). From a simplified version balance of payments identities, the \( CAB_t \) equals the acquisition of Net Foreign Assets \( (NFA_t) \) less the incurrence of Net Foreign Liabilities. Assets and liabilities are recorded end-of-period.

\[
CAB_t = \Delta NFA_t - \Delta NFL_t
\]

(9)

The change in net foreign liabilities can be decomposed into net inflows of foreign direct investment \( (FDI_t) \), as well as the accumulation of total external debt \( D_t \) (portfolio liabilities, public and private). For simplicity, we assume no changes in the stock of net foreign assets, which is a benign assumption for most developing countries.

\[
\Delta NFL_t = FDI_t + (D_t - D_{t-1}) \quad \Delta NFA_t \approx 0
\]

Substituting into Equation (9), and dividing by GDP \( (Y_t) \) and using \( Y_t/Y_{t-1} = (1 + g_{pc}^{FE})(1 + g_{NA,t}) \), one can write the \( CAB_t/Y_t \) as:

\[
\frac{CAB_t}{Y_t} = -\left[ \frac{D_t}{Y_t} - \frac{D_{t-1}/Y_{t-1}}{(1 + g_{pc}^{FE})(1 + g_{NA,t})} \right] - \frac{FDI_t}{Y_t}
\]

(10)

Combining Equations (8) and (10) one can relate savings and investment with an external debt constraint.

\[
\frac{I_t}{Y_t} = \frac{S_t}{Y_t} + \frac{FDI_t}{Y_t} + \left[ \frac{D_t}{Y_t} - \frac{D_{t-1}/Y_{t-1}}{(1 + g_{pc}^{FE})(1 + g_{NA,t})} \right]
\]

(11)

This can also be re written in terms of required savings.

\[
\frac{S_t}{Y_t} = \left[ \frac{I_t}{Y_t} - \frac{FDI_t}{Y_t} \right] - \left[ \frac{D_t}{Y_t} - \frac{D_{t-1}/Y_{t-1}}{(1 + g_{pc}^{FE})(1 + g_{NA,t})} \right]
\]

(12)

- **Model 3 (\( D/Y \) constraint):** Assume a path for external debt-to-GDP \( (D_t/Y_t) \), and net inflows of Foreign Debt Investment \( FDI_t/Y_t \), and use Equation (11) to calculate \( I_t/Y_t \) for a given level of savings \( (S_t/Y_t) \). Then use Step 1-4 from Model 1 (Section 1.3) to calculate growth.

- **Model 1 and 2 (\( D/Y \) constraint):** Combine paths for \( D_t/Y_t \) and \( FDI_t/Y_t \) with path for \( I_t/Y_t \) assumed (Model 1) or implied (Model 2) to generate implied savings using Equation (12).

The effect of FDI on required national savings: With an external debt constraint, an increase in \( FDI_t/Y_t \) acts as a substitute for national savings \( (S_t/Y_t) \). That is, in Equation (12), national savings only have to cover the fraction of investment not funded by FDI \( (I_t/Y_t - FDI_t/Y_t) \), rather than the whole amount.

\[ \text{Initial values of } CAB_t/Y_t \text{ are taken from the MFMod Database and WDI. The World Development Indicators are the source of } FDI_t/Y_t \text{ (code: BX.KLT.DNIF.1Y.DD.GD.ZS.) and } D_t/Y_t \text{ (calculated as DT.DOD.DECT.CD ÷ NY.GDP.MKTP.CD).} \]

\[ \text{For economies not open to capital flows, assume } CAB_t/Y_t \approx 0. \]
4 Understanding the drivers of growth

To understand the drivers of growth, in this section we take log-linear approximation to simplify the formulas. Note that quantitative analysis should be done using the exact equations above because even small approximation errors can compound over time. First combine Equation 2 and 3:

\[
1 + g_{pc,y,t+1} = [1 + g_{ω,t+1}] [1 + g_{ϕ,t+1}] [(1 + g_{A,t+1}) [1 + g_{k,t+1}]^{1-\beta} [1 + g_{h,t+1}]^\beta
\]

Taking logs, and using the approximation \(\ln(1 + x) \approx x\) (for small \(x\)) this becomes

\[
g_{pc,y,t+1} \approx g_{A,t+1} + g_{ω,t+1} + g_{ϕ,t+1} + (1 - \beta) g_{k,t+1} + \beta g_{h,t+1}
\]

(13)

Taking logs of the capital-per-worker growth Equation 5, yields:

\[
\ln(1 + g_{k,t+1}) = \ln \left[ 1 + \frac{I_t}{Y_t} \right] - \ln(1 + g_{N,t+1}) - \ln(1 + g_{ϕ,t+1}) - \ln(1 + g_{ω,t+1})
\]

Applying the \(\ln(1 + x) \approx x\) approximation (for small \(x\)):

\[
g_{k,t+1} \approx \frac{I_t}{Y_t} - \delta - g_{N,t+1} - g_{ϕ,t+1} - g_{ω,t+1}
\]

(14)

Combining Equation 13 and 14 gives the approximate determinants of growth:

\[
g_{pc,y,t+1} \approx g_{A,t+1} + \beta (g_{ω,t+1} + g_{ϕ,t+1} + g_{h,t+1}) + (1 - \beta) \left[ \frac{I_t}{Y_t} - \delta - g_{N,t+1} \right]
\]

(15)

The effect of most factors (except TFP) on growth depends on the labor share \(\beta\), which is around 0.5 on average across all countries (PWT 8.1, 9.0 and 9.1).

- TFP growth \((g_{A,t+1})\) has the largest direct effect on growth: a 1ppt increase in TFP growth increases growth by 1ppt (regardless of \(\beta\)).

- A 1ppt increase human capital growth \((g_{ϕ,t+1})\), labor force participation rate growth \((g_{ω,t+1})\) or working age population share growth \((g_{ω,t+1})\) increase per capita GDP growth by \(\beta\)ppt. If \(\beta \approx 0.5\), then a 1ppt increase in each of these factors, has half the effect as a 1ppt increase in TFP growth.

- Population growth \((g_{N,t+1})\) and depreciation \((\delta)\) reduce per capita GDP growth by \(1 - \beta\), because they reduce capital depth (capital per worker) by either reducing the amount of capital \((\delta)\) or increasing the number of workers \((g_{N,t+1})\).

- The effect of an increase in the the investment share of GDP depends on both the capital share \((1 - \beta)\), as well as the existing capital-to-output ratio \((K/Y)\). Assuming \(1 - \beta = 0.5\), a large 10ppt increase in the investment share of GDP (eg from 20% to 30%), raises growth by 2.5ppt per year if \(K/Y = 2\) (i.e 0.1 x 0.5/2), but only 1.25ppt if \(K/Y = 4\).

  - This means that a investment-led growth strategy will quickly become less effective, unless it is accompanied by other reforms to boost productivity, human capital or participation to mitigate the increase in \(K/Y\).

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*In standard growth account exercises for OECD countries like the US, the labor share is around 2/3.*
4.1 Relation to the Incremental Capital to Output Ratio (ICOR)

Many countries use the Incremental Capital to Output Ratio (ICOR) to analyze the effectiveness of investment in boosting growth. Specifically, the (gross) marginal ICOR is the percentage point increase in the investment share of GDP needed to boost headline GDP growth by 1%.

\[ g_Y \approx \frac{1}{ICOR} \frac{I}{Y} \]

where \( g_{Y,t+1} = \frac{Y_{t+1} - Y_t}{Y_t} \approx g_{pc,t+1} + g_{N,t+1} \) is the growth rate of headline GDP (not per capita). As the relationship in Equation 16 is assumed to be proportional, the average ICOR \((ICOR_a)\) is equal to the marginal ICOR \((ICOR_m)\) and so is just called the ICOR. Take two examples: (a) if the marginal ICOR is 4.3, then if the country wants to increase headline GDP growth by 1%, it must increase the investment share of GDP by 4.3 ppt; (b) if the average ICOR is 4.3, an 8% target GDP growth rate can be achieved with an investment share of 34.4% of GDP \((I/Y = ICOR_a \times g_Y = 4.3 \times 8\% = 34.4\%)\), rearranging Equation 16.

In the long-term growth model (LTGM), the approximate relationship between GDP growth and \(I/Y\) is linear but not proportional, which means that the ICOR only applies to an analysis of extra units of growth or investment\(^{10}\). Rearranging Equation 15,

\[ g_{Y,t+1} \approx \beta (g_{N,t+1} + g_{A,t+1} + g_{g,t+1} + g_{h,t+1}) - (1 - \beta) \delta + \frac{1}{ICOR_a} \omega, t + 1 \]

In the LTGM, to boost GDP growth by 1%, one must raise investment share of GDP by:

\[ ICOR_{LTGM} = \frac{1}{1 - \beta} \frac{K_t}{Y_t} \]

For example, suppose \( \beta = 0.5, \) and \( K_0/Y_0 = 2.2 \) then the marginal ICOR is 4.4, so one needs to increase the investment share of GDP by 4.4ppt to boost headline GDP growth by 1%.

Critically, the marginal ICOR is not constant over time. A investment-led growth strategy not accompanied by growth in TFP etc will rapidly increase \( K/Y \), leading to an increase in the ICOR. This makes future investment less effective in boosting growth. The level of growth, and the rate of \( K/Y \) growth also depend on the terms in the intercept.

Readers will note that the inverse of the marginal ICOR is just the marginal product of capital \((MPK)\):

\[ MPK = \frac{\partial Y}{\partial K_t} = \frac{1 - \beta}{K_t} \frac{Y_t}{1/ICOR_m} \]

The net return on capital is \( R = MPK - \delta \). In the example above, if the marginal ICOR is 4.4, then the marginal product of capital is 4.4\(^{-1}\) = 23\%. With \( \delta = 5\% \) the net return on capital is 23\%-5\%-18\% per year. A rising marginal ICOR as \( K/Y \) increases during an investment-led growth program is the same as saying that the return to capital is falling.

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\(^9\)As the name suggests, the Incremental Capital to Output ratio originally referred to net investment: \( \frac{\Delta K_t}{\Delta Y_t} = \frac{I^{net}}{Y_{t-1}} \). However, in practice net investment was hard to measure in developing countries due to a lack of data on depreciation rates and capital stocks. As result many analysts approximated net investment with gross investment, giving the rule-of-thumb measure presented in Equation 16. It should be noted that the rule-of-thumb gross ICOR is not the same as the original net ICOR: the net ICOR will be smaller by \( \frac{\Delta K_{t-1}}{\Delta Y_{t-1}} \), which could be important if growth rates are small. For example, with \( K/Y = 2 \) and \( \delta = 0.05 \), the net ICOR is smaller by \( 5\% \) if GDP growth is \( 2\% \), but only by \( 2\% \) if GDP growth is \( 5\% \).

\(^{10}\)That is, the marginal ICOR and the average ICOR will be different. The average ICOR will change with the investment rate.
5 Poverty Extension: the effect of growth on the poverty rate

The poverty rate is determined by the distribution of per capita income, as well as its average level. LTGM v4 introduces the income distribution by assuming that the natural log of income per capita follows a normal distribution, \( \ln(y^{pc}) \sim N(\mu, \sigma^2) \). The log-normal income distribution greatly simplifies the calculation of poverty rates, needs very few parameters and little data and empirically is a good approximation of the majority of the income distribution (see Lopez and Servén 2006 and Bourguignon 2007). The addition of the poverty extension does not substantially affect the workings of the Model 1-3 above.

The poverty headcount rate \( P \) is the proportion of people with incomes below the poverty line \( L \). Combined with the mean \( \mu \) and standard deviation \( \sigma \) of the underlying normal distribution (not the mean/SD the actual income distribution), the poverty rate can be calculated from the standard normal cumulative density function (CDF) as in Equation 19:

\[
P_t = \Phi\left(\frac{\ln L - \mu_t}{\sigma_t}\right)
\] (19)

For a log normal distribution, the Gini coefficient of income inequality \( G \), is a transformation of standard deviation \( \sigma \) of the underlying normal distribution:

\[
G_t = 2\Phi\left(\frac{\sigma_t}{\sqrt{2}}\right) - 1
\] (20)

To calculate \( \mu \) from data on \( P \) and \( L \) invert Equation 19:

\[
\mu_t = \ln L - \sigma_t \Phi^{-1}(P_t)
\] (21)

where \( \sigma \) can be calculated from \( G \) by inverting Equation and 20:

\[
\sigma_t = \sqrt{2} \times \Phi^{-1}\left(\frac{G_t + 1}{2}\right)
\] (22)

The mean income (GDP per capita) is given by \( \exp(\mu + \sigma^2/2) \). Absent any change in income inequality (i.e. with constant \( G \) and \( \sigma \)), economic growth shifts the whole income distribution to the right (proportionately) by increasing \( \mu \). Allowing for a change in inequality, the per capita growth rate is given by Equation 23:

\[
1 + g_{y,t+1}^{pc} = \frac{\bar{y}_{t+1}^{pc}/\bar{y}_t^{pc}}{\exp(\mu_{t+1} + \sigma_{t+1}^2/2)} = \frac{\exp(\mu_{t+1} + \sigma_{t+1}^2/2)}{\exp(\mu_t + \sigma_t^2/2)} = \exp(\mu_{t+1} - \mu_t + \frac{1}{2}(\sigma_{t+1}^2 - \sigma_t^2))
\] (23)

We can rewrite this as Equation 24, which is used update the mean of the underlying normal distribution in Models 1 and 3.

\[
\mu_{t+1} = \ln(1 + g_{y,t+1}^{pc}) + \mu_t - \frac{1}{2}(\sigma_{t+1}^2 - \sigma_t^2)
\] (24)

Using the approximation \( \ln(1 + g) \approx g \) (for small \( g \)), this becomes: \( \mu_{t+1} \approx g_{y,t+1}^{pc} + \mu_t - \frac{1}{2}(\sigma_{t+1}^2 - \sigma_t^2) \) which suggests that with constant income inequality \( (\sigma_{t+1}^2 = \sigma_t^2) \), an extra percentage point of per capita GDP growth increases \( \mu \) by one percentage point.

Steps to solve for poverty rates in Model 1, 2 and 3 using the log-normal distribution

In models 1 and 3, growth fundamentals (investment savings etc) determine the path of per capita growth \( \{g_{y,t+1}^{pc}\} \), from which we calculate the change in the poverty rate. The steps are as follows:

1. Assume a path for the Gini coefficient on income \( \{G_t\} \) from the first period until the end of the simulation (this could be constant), and then calculate \( \sigma_t \) for each year using Equation 22.\(^{13}\)

\(^{11}\)Thanks to Aart Kraay for suggesting this approach. Although the distribution of income is always log normal, \( \mu \) and \( \sigma \) vary across countries and over time.

\(^{12}\)The CDF (proportion less than \( x \)) is \( Pr(X \leq x) = \Phi(x) \), which is \texttt{normsdist}(x) in Excel. The inverse function \( \Phi^{-1}(Pr) \), is \texttt{normsinv}(Pr) in Excel.

\(^{13}\)This can be done in Excel as \( \sigma_t = \sqrt{2} \times \texttt{NORMSINV}(0.5 \times (G_t + 1)) \)
2. Choose an initial poverty line \( L \) and initial headcount poverty rate \( P_t \) at that poverty line. The pre-loaded defaults are the international extreme poverty line ($1.90/day 2011 PPP), and the extreme poverty rate for the most recent household survey in PovcalNet. Alternatively, users can manually enter their own \( \{P, L\} \) for the own national poverty line (which is often quite different)\(^{14}\) Calculate the initial \( \mu_t \) using Equation \( 21 \)\(^{15}\)

3. For each period after the first, update \( \mu_{t+1} \) using Equation \( 24 \)

4. For each period after the first, calculate the poverty rate \( P_{t+1} \) using Equation \( 19 \) (given \( \mu_{t+1}, \sigma_{t+1} \) and \( L \))\(^{16}\)

**Growth Elasticity of Poverty (GEP) — understanding the effect of growth on poverty**

In the literature, the *growth elasticity of poverty (GEP)* \( \varepsilon_p \) is the percentage (not percentage point) fall in the headcount poverty rate from a 1% increase in per capita income (i.e. from a 1% per capita growth rate). For a log normal distribution, the GEP is given by Equation \( 25 \) which helps the LTGM user to compare their estimates with those in the empirical literature.\(^{17}\) Bourguignon (2007) reports empirical estimates of about 1.5 across poverty spells, though earlier papers estimate the GEP of 2 or even 3.

However, the GEP varies substantially across countries and over time. With a log normal income distribution, the GEP is mechanically higher for countries with low rates of poverty, because even a small change in the poverty headcount is large as a percentage of a very small base. The GEP is also higher in countries that are more equal (a smaller Gini coefficient of income). As noted by Bourguignon (2007) this means that a reduction in inequality has a “double dividend”: first it reduces poverty in and of itself, and second it increases the growth elasticity of poverty.

\[
\varepsilon_p \equiv -\frac{\partial \ln P_t}{\partial \ln y_t} = -\frac{\partial P_t}{\partial \mu_t} \frac{1}{\sigma_t} = \frac{1}{\sigma_t} \phi\left(\frac{\ln L - \mu_t}{\sigma_t}\right)\tag{25}
\]

A closely related metric is *growth semi-elasticity of poverty*, which we define as the *percentage point* change in poverty for an extra 1% increase in per capita income (i.e. from a 1% per capita growth rate) as in Equation \( 26 \)

\[
\Delta_p \equiv -\frac{\partial P_t}{\partial \ln y_t} = \varepsilon_p \times P_t = \frac{1}{\sigma_t} \phi\left(\frac{\ln L - \mu_t}{\sigma_t}\right)\tag{26}
\]

The growth semi-elasticity of poverty is inverse-U shaped in the poverty rate, with the largest response of poverty rates in *percentage points* generally occurring in countries with a poverty rate of around 0.5. At this point there are many people just below the poverty line who can be moved out of poverty by a small increase in income. The *growth semi-elasticity of poverty* is calculated as a memorandum item in the LTGM spreadsheets.

**GEP Implementation in the LTGM Spreadsheet** In the LTGM, users can input their own GEP instead of using the one implied by the log-normal income distribution (Equation \( 25 \)). For Model 1, Model 3, and Model 2 (with a non-poverty target), the poverty rate is given by calculating:

\[
P_{t+1} = (1 - \varepsilon_p \times g_{y,t+1}^{pc}) P_t \tag{27}
\]

Alternatively, in Model 2 (with a poverty target) required growth is given by:

\[
g_{y,t+1}^{pc} = -\left(P_{t+1}/P_t - 1\right)/\varepsilon_p\tag{28}
\]

\(^{14}\)It is only the initial poverty rate \( P \) that affects the model. Changes in \( L \) (for example changing from per-day to per-month, or the currency of measurement), do not affect the model as \( \mu \) scales accordingly.

\(^{15}\)This can be done in Excel as \( \mu_t = \ln(L_t) - \sigma_t \times \text{NORMSINV}(P_{t+1}) \)

\(^{16}\)This can be done in Excel as \( P_{t+1} = \text{NORMDIST}((\ln(L) - \mu_{t+1})/\sigma_{t+1}) \)

\(^{17}\)This can be done in Excel as \( P_{t+1} = \text{NORMDIST}((\ln(L) - \mu_{t+1})/\sigma_{t+1}) \)

\(^{18}\)An increase in average income (keeping \( \sigma \) constant) always reduces the poverty rate, so we follow Bourguignon (2007) and make the elasticity positive by pre-multiplying by -1. The second equality in Equation \( 25 \) comes from the mean of a log-normal distribution \( \ln y_t = \mu_t + \sigma_t^2/2 \) (keeping \( \sigma_t \) constant) and the third equality comes from applying Leibniz’s rule to Equation \( 19 \). This equation is similar to Equation 3’ in Bourguignon (2007). Here \( \phi(.) \) is the normal probability distribution function (in Excel \( \text{NORMDIST}(x, 0, 1, \text{FALSE}) \) and Equation \( 25 \) is \( (1/\sigma_{t+1}) \times \text{NORMDIST}((\ln(L) - \mu_{t+1})/\sigma_{t+1}, 0, 1, \text{FALSE})/\text{NORMDIST}((\ln(L) - \mu_{t+1})/\sigma_{t+1}) \).
Shared Prosperity Premium (SPP)

One of the goals of the World Bank is to “Promote shared prosperity by fostering the income growth of the bottom 40% for every country.”

In the LTGM poverty extension, the average income of the bottom 40% of the population is given by Equation (29) where \( k_t \equiv \exp(\sigma_t \Phi^{-1}(0.4) + \mu_t) \) is the income cutoff that defines the bottom 40% (which changes over time and across countries).\(^\text{18}\)

\[
E(y^{pc}|y^{pc} < k_t) = 0.4^{-1} e^{\mu_t + \sigma_t^2/2} \Phi(\Phi^{-1}(0.4) - \sigma_t)
\]  

(29)

The income share of the bottom 40% of the population (\(SB40\)) can be expressed as\(^\text{18}\)

\[
SB40_t \equiv \frac{E(y^{pc}|y^{pc} < k_t) \times 0.4}{E(y^{pc})} = \frac{0.4^{-1} e^{\mu_t + \sigma_t^2/2} \Phi(\Phi^{-1}(0.4) - \sigma_t) \times 0.4}{e^{\mu_t + \sigma_t^2/2}} = \Phi(\Phi^{-1}(0.4) - \sigma_t)
\]  

(30)

In terms of the Gini coefficient (using Equation (22)) the income share of the bottom 40% can be written as:

\[
SB40_t = \Phi(\Phi^{-1}(0.4) - \sqrt{2} \times \Phi^{-1}((G_t + 1)/2))
\]  

(31)

As such, the growth rate of average income of the bottom 40% is defined as the average income gross growth rate \(1 + g_{y,t+1}^{pc}\) times the gross growth rate of the income share of the bottom 40% \((SB40_{t+1}/SB40_t)\):

\[
1 + g_{40,t+1} = \frac{E(y^{pc}|y^{pc} < k_{t+1})}{E(y^{pc}|y^{pc} < k_t)} = (1 + g_{y,t+1}^{pc}) \frac{\Phi(\Phi^{-1}(0.4) - \sigma_{t+1})}{\Phi(\Phi^{-1}(0.4) - \sigma_t)}
\]  

(32)

where \(g_{y,t+1}^{pc}\) is the economy-wide per capita growth rate as defined in Equation (23) and \(\sigma\) is the SD of the underlying normal distribution (which is a one-to-one transformation of the Gini coefficient by Equation (27)).

The Shared Prosperity Premium (SPP) (Equation (33)) is the excess income growth of the bottom 40% \((g_{40,t+1})\) over the average per capita growth rate of the whole economy \((g_{y,t+1}^{pc})\):

\[
SPP_{t+1} \equiv \ln(1 + g_{40,t+1}) - \ln(1 + g_{y,t+1}^{pc}) \approx g_{40,t+1} - g_{y,t+1}^{pc}
\]  

(33)

Combining Equations (33) and (32) one can see gain an expression for the SPP\(_{t+1}\), which is just the log growth rate of income share of the bottom 40%.

\[
SPP_{t+1} = \ln \left[ \frac{\Phi(\Phi^{-1}(0.4) - \sigma_{t+1})}{\Phi(\Phi^{-1}(0.4) - \sigma_t)} \right] = \ln \left[ \frac{SB40_{t+1}}{SB40_t} \right]
\]  

(34)

(35)

From Equation (34) one can see that when there is no change in income inequality (such that \(\sigma_{t+1} = \sigma_t\) and \(G_{t+1} = G_t\)), then the shared prosperity premium will be zero, and the growth rate of the incomes of the bottom 40% will be the same as the per capita growth rate of the economy as a whole (recall from equation (22) that \(\sigma_t = \sqrt{2} \times \Phi^{-1}((G_t + 1)/2)\)). As such, if income follows a log-normal distribution then there is almost a one-to-one relationship between the change in inequality (as measured by the Gini coefficient) and the shared prosperity premium: a fall (raise) in inequality is equivalent to a positive (negative) shared prosperity premium.

\(^\text{18}\)This follows from the expression for a conditional mean of a log normal distribution:  
\[E(X|X < k_t) = e^{\mu + \sigma^2/2} \Phi_{\mu_t, \sigma_t^2} \Phi^{-1}(0.4) / \Phi(\Phi^{-1}(0.4) - \mu_t)\], where \(\Phi(\Phi^{-1}(0.4) - \mu_t) = 0.4\)

\(^\text{19}\)To see this, we normalize the population to 1, which means that the mean per capita income \(y^{pc}\) equals total income.
Shared Prosperity Implementation in the LTGM Spreadsheet

Given Equation \(34\), the Shared Prosperity Premium enters the LTGM as an alternative way for the user to specify the path of inequality, or to summarize the implications for the bottom 40% of a given path of Gini coefficient. As such, the shared prosperity mostly enters in the sheet InputdataA when the user is specifying the path for inequality (and the SPP is plotted in GraphsA)\(^{20}\).

- If the user specifies a path for the Gini coefficient, then the implied Shared Prosperity Premium is calculated using Equation \(34\) as a residual (using Equation \(22\) to substitute out for the Gini as an intermediate step).
- If the user specifies a path for the Shared Prosperity Premium (SPP):
  - the user must still specify an initial Gini coefficient \(G_t\) for the first year, which is then converted into an initial \(\sigma_t\) using Equation \(22\).
  - \(\sigma_{t+1}\) given by Equation \(36\) where a higher SPP increases reduces \(\sigma_{t+1}\)
    \[
    \sigma_{t+1} = \Phi^{-1}(0.4) - \Phi^{-1}\left[e^{SPP_{t+1}}\Phi(\Phi^{-1}(0.4) - \sigma_t)\right]
    \] (36)
  - the Gini coefficient \(G_{t+1}\) (which enters the model spreadsheets) is calculated using Equation \(20\).

• The income share of bottom 40% of the population \((SB40)\) is recorded as a memorandum item using Equation \(30\).

References


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\(^{20}\)The income growth of the bottom 40% in each of Model 1/1s/2/2s/3/3s is also summarized at a poverty memorandum item at the bottom of each of those sheets and plotted in GraphsB.